II. METRIC-CONNECTION THEORIES AS THE GAUGE THEORIES OF SPACETIME SYMMETRIES

1. Fibre Bundle Description of Gauge Theories

In this chapter I wish to develop an analogy between the metric-connection theories of gravity and the gauge theories of elementary particle interactions. In order to make this analogy, I need to give an appropriate description of the gauge theories. In this section I give such a description in the highly mathematical language of fibre bundles and explain it in the more mundane language of the physicist.

In the fibre bundle language, a gauge theory with gauge group, G, prescribes the following geometrical objects:

a. a principal G-bundle, P, over spacetime M;

b. a G-vector bundle, E, associated to P with representation, R, acting on the typical fibre, V;

c. a global cross section, ψ, of the bundle, E; and

d. a connection, A, on the bundles, P and E.

A particular gauge theory is specified by giving the gauge group, G, the representation, R, and an action functional, $S[\psi,A]$. A classical solution of this theory is any choice of the bundles, P and E, together with a cross section, ψ, and a connection, A, which make the action stationary.

What does all this mathematics mean? The rigorous definitions appear in Appendix B. However, I explain it here by giving an example. Suppose one wants to consider a gauge theory of the weak (and electromagnetic) interactions in which the leptons and quarks have 16 flavors and the gauge group is U(16). In a particular choice of gauge, the lepton wave function, χ, is just 16 complex functions on spacetime, M.

(I am ignoring the spinor nature of the lepton wave function.) It is convenient to indicate the gauge by a label, α, and to count the components by an index, k = 1...16. Thus the lepton wave function is $\overset{\alpha k}{\chi}$ and at a point, $p \in M$, its value, $\overset{\alpha k}{\chi}(p)$, is an element of C^{16}. Under a change of gauge, these 16 numbers mix according to a U(16) transformation, $g_{\alpha\beta}(p)$:

$$\overset{\alpha k}{\chi}(p) = [g_{\alpha\beta}(p)]^{k}_{\ j} \overset{\beta j}{\chi}(p). \tag{1}$$

The gauge transformation is global if $g_{\alpha\beta}(p)$ is independent of p; otherwise it is local.

Is there a gauge invariant way of describing the lepton wave function? Yes! Simply put an abstract 16 dimensional complex vector space, F_p, at each point, p. The value of the lepton wave function, χ, at a point, p, is a vector, $\chi(p)$, in the space, F_p. Making a choice of gauge (say the α-gauge) simply corresponds to making a choice of basis (say $\overset{\alpha}{e}_k(p)$) in each space, F_p. The lepton wave function at p may be expanded in this basis as

$$\chi(p) = \overset{\alpha}{e}_k(p) \overset{\alpha k}{\chi}(p) \ , \tag{2}$$

and the components, $\overset{\alpha k}{\chi}(p)$, are the description of $\chi(p)$ in the α-gauge. Upon making the gauge transformation (1), the basis is also changed according to

$$\overset{\alpha}{e}_k(p) = \overset{\beta}{e}_j(p) \ [g^{-1}_{\alpha\beta}(p)]^{j}_{\ k} \ , \tag{3}$$

so that $\chi(p)$ in (2) remains invariant.

But there are too many bases! If one were to allow arbitrary bases, then under a change of bases the components of $\chi(p)$ could change by a GL(16,C) transformation rather than just a U(16) transformation. The

easiest way to place a restriction on the choice of bases so that they are always related by a U(16) transformation, is to assume that there is an hermitian form, ϕ, on each vector space, F_p, and to require the frames, $\overset{\alpha}{e}_k$, to be orthonormal relative to ϕ. Thus

$$\overset{\alpha}{\phi}_{jk}(p) = \phi(\overset{\alpha}{e}_j(p) , \overset{\alpha}{e}_k(p)) = \delta_{jk}. \tag{4}$$

Two such frames are always related by a U(16) transformation, $g_{\alpha\beta}(p)$, since

$$\delta_{mn} = \phi(\overset{\beta}{e}_m(p) , \overset{\beta}{e}_n(p)) = [g_{\alpha\beta}(p)]^{j}{}_{m}^{*} \, [g_{\alpha\beta}(p)]^{k}{}_{n} \, \delta_{jk}. \tag{5}$$

(If the gauge group is other than U(16), then other or additional restrictions must be placed on the class of admissible frames.)

Of course the various vector spaces, F_p, must be glued together so that it makes sense to say that the wave function. $\chi(p)$, is continuous or differentiable in p. The union, B, of all the vector spaces, F_p, with a suitable topology is an example of a U(16)-vector bundle. The wave function, χ, may be regarded as a map, $\chi: M \to B$, such that $\chi(p) \in F_p \subset B$. Such a map is called a global cross section of the bundle, B.

The set, P, of all orthonormal frames, $\overset{\alpha}{e}_k(p)$, at all points $p \in M$, with a topology induced by that of B, is an example of a principal U(16) - bundle. A local orthonormal frame field for the bundle, B, may be regarded as a map, $\overset{\alpha}{e}_k : U \to P$, such that $\overset{\alpha}{e}_k(p)$ is an orthonormal basis for F_p. Such a map is called a local cross section of the bundle, P.

The bundles, P and B, are called associated in that a change of gauge (or orthonormal frame field) in the bundle, B, corresponds to a change of gauge (or local cross section) in the bundle, P. (The rigorous definitions of G-vector bundles, principal G-bundles, associated G-bundles, and cross sections appear in Appendix B.)

How does this discussion of a U(16) gauge theory relate to the general fibre bundle description of a gauge theory at the beginning of this section? The orthonormal frame bundle, P, is the principal G-bundle mentioned in that description. What are E, ψ and A?

The cross section, ψ, is the vector field whose components are the components of the wave functions of all of the source fields. (I use the term matter field to denote any dynamic field on spacetime other than the spacetime metric, the spacetime connection, and any other gravitational field. I use the term source field to denote any matter field other than the gauge potentials. Thus the source fields include the leptons, the quarks, the Goldstone-Higgs fields, etc.) In a particular choice of gauge, the vector field, ψ, has values in a vector space, V, and under a gauge transformation it transforms according to a representation, R, which is the direct sum of the representations for all the source fields. Putting a copy, V_p, of the vector space, V, at each point, $p \in M$, one obtains the G-vector bundle, E. Notice that the bundle, B, is a sub-vector bundle of the bundle, E, and the lepton wave function, χ, is the projection of the cross section, ψ, into the subbundle, B. Further, the bundles, P, B and E, are all associated in that a choice (or change) of gauge in one of them corresponds to a choice (or change) of gauge in the others.

Finally, the connection, A, describes the gauge potentials. As discussed in Appendix B, in each gauge (say the α-gauge) the connection determines a 1-form, $\overset{\alpha}{A}$, with values in $\mathcal{L}G$, the Lie algebra of G. This 1-form can be expanded in a coordinate basis, dx^a, for the 1-forms and a dimensionless basis, T_p, for $\mathcal{L}G$:

$$\overset{\alpha}{A} = \overset{\alpha}{A}{}^{p}_{a}\, T_p\, dx^a \,. \qquad (6)$$

The components, $\overset{\alpha}{A}{}^{P}{}_{a}$, are the gauge potentials in the α-gauge. Under the gauge transformation (3), the gauge potentials transform according to

$$\overset{\alpha}{A}{}^{P}{}_{a}\, T_P = g_{\alpha\beta}\, \overset{\beta}{A}{}^{P}{}_{a}\, T_P\, g^{-1}_{\alpha\beta} + g_{\alpha\beta}\, \partial_a(g^{-1}_{\alpha\beta}) \ . \tag{7}$$

From the gauge potentials, $\overset{\alpha}{A}{}^{P}{}_{a}$, one defines the gauge fields,

$$\overset{\alpha}{F}{}^{P}{}_{ab} = \partial_a \overset{\alpha}{A}{}^{P}{}_{b} - \partial_b \overset{\alpha}{A}{}^{P}{}_{a} + f^{P}{}_{QR}\, \overset{\alpha}{A}{}^{Q}{}_{a}\, \overset{\alpha}{A}{}^{R}{}_{b}, \tag{8}$$

where the $f^{P}{}_{QR}$ are the dimensionless structure constants of $\mathcal{L}G$ defined by

$$[T_Q, T_R] = f^{P}{}_{QR}\, T^{P}. \tag{9}$$

The curvature is then the $\mathcal{L}G$ valued 2-form

$$\overset{\alpha}{F} = \overset{\alpha}{F}{}^{P}{}_{ab}\, T_P\, dx^a \wedge dx^b. \tag{10}$$

These transform covariantly according to

$$\overset{\alpha}{F} = g_{\alpha\beta}\, \overset{\beta}{F}\, g^{-1}_{\alpha\beta} \ , \tag{11}$$

and

$$\overset{\alpha}{F}{}^{P}{}_{ab} = \mathrm{ad}(g_{\alpha\beta})^{P}{}_{Q}\, \overset{\beta}{F}{}^{Q}{}_{ab} \ , \tag{12}$$

where ad denotes the adjoint representation of G acting on $\mathcal{L}G$.

Returning to the U(16) example, the connection induces a covariant derivative on the vector bundles, B and E. Since the cross section, χ, of the bundle, B, transforms as in (1) according to the defining representation of U(16), its covariant derivative is

$$\nabla_a \overset{\alpha}{\chi}{}^{k} = \partial_a \overset{\alpha}{\chi}{}^{k} + \overset{\alpha}{A}{}^{P}{}_{a} (T_P)^{k}{}_{j}\, \overset{\alpha}{\chi}{}^{j} \ . \tag{13}$$

Similarly, the cross section, ψ, of the bundle, E, transforms according to the representation, R:

$$\overset{\alpha}{\psi}{}^{k} = (Rg_{\alpha\beta})^{k}{}_{j}\ \overset{\beta}{\psi}{}^{j}, \tag{14}$$

where now j, k = 1...N. So its covariant derivative is

$$\nabla_a \overset{\alpha}{\psi}{}^{k} = \partial_a \overset{\alpha}{\psi}{}^{k} + \overset{\alpha}{A}{}^{P}{}_{a} (RT_P)^{k}{}_{j}\ \overset{\alpha}{\psi}{}^{j}. \tag{15}$$

The gauge potentials are often called compensating fields because the non-covariant second term in their transformation law (7) exactly cancels the non-covariant term in the transformation of the partial derivative $\partial_a \overset{\alpha}{\chi}{}^{k}$ in $\nabla_a \overset{\alpha}{\chi}{}^{k}$ (or $\partial_a \overset{\alpha}{\psi}{}^{k}$ in $\nabla_a \overset{\alpha}{\psi}{}^{k}$). Thus the covariant derivatives transform covariantly:

$$\nabla_a \overset{\alpha}{\chi}{}^{k} = (g_{\alpha\beta})^{k}{}_{j}\ \nabla_a \overset{\beta}{\chi}{}^{j}, \tag{16}$$

$$\nabla_a \overset{\alpha}{\psi}{}^{k} = (Rg_{\alpha\beta})^{k}{}_{j}\ \nabla_a \overset{\beta}{\psi}{}^{j}. \tag{17}$$

From now on I work primarily in a single gauge; so I will no longer write the gauge label, α, unless necessary.

Notice that in equations (8), (13) and (15), I have chosen $f^{P}{}_{QR}$, T_P, and RT_p to be dimensionless. Hence, $A^{P}{}_{a}$ has the dimensions $(\text{length})^{-1}$ and $F^{P}{}_{ab}$ has the dimensions $(\text{length})^{-2}$. In the case of electromagnetism, whose gauge group is U(1), there is only one generator, T_o, which may be chosen as $T_o = i$ in the defining representation and is then represented by $RT_o = n\ i$ in the complex representation with electric charge $q = n\ e$. In that case, $A^{o}{}_{a}$ and $F^{o}{}_{ab}$ are related to the conventional electromagnetic potential, A^{conv}_{a}, and field, F^{conv}_{ab}, by

$$A^{o}{}_{a} = \frac{e}{\hbar c} A^{conv}_{a}, \qquad F^{o}{}_{ab} = \frac{e}{\hbar c} F^{conv}_{ab}. \tag{18}$$

Again considering the general gauge theory, the gauge potentials, A^P_a, and the source field, ψ, are used to construct the action functional, $S[\psi,A]$, which must be invariant under gauge transformations. It is usually assumed that the action is local; i.e. that it may be written as

$$S[\psi,A] = \int L_M \sqrt{-\tilde{g}}\, d^4x \,, \tag{19}$$

where the matter Lagrangian,

$$L_M = L_M(\psi, \partial\psi, \ldots, \partial^{(m)}\psi, A, \partial A, \ldots, \partial^{(n)}A), \tag{20}$$

is a strictly local function of ψ, A, and a finite number of their derivatives. (A function, $f = f(g_i)$, is a strictly local function of the functions, g_i, if the value of f at a point, x, only depends on the values of the g_i's at the point, x.) It is also usually assumed that the Lagrangian, L_M, is a scalar under gauge transformations. I make both of these assumptions throughout the thesis.

The matter Lagrangian for the hypothetical U(16) theory might contain the term,

$$-\frac{\hbar c}{16\pi g^2}\, \gamma_{PQ}\, F^P_{ab}\, F^{Qab} \,, \tag{21}$$

as a kinetic Lagrangian for the gauge fields, and the term,

$$\frac{\hbar c}{4\pi}\, \phi_{jk}\, \bar{\chi}^j (i\, \gamma^\mu\, \nabla_\mu - \frac{mc}{\hbar})\, \chi^k, \tag{22}$$

to describe the leptons. Notice that in each of these terms there is an inner product of some form: γ_{PQ} is a group metric; i.e. a symmetric bilinear form on the Lie algebra. ϕ_{jk} is the hermitian form mentioned above equation (4).

It is usually assumed that these inner products are invariant under gauge transformations; i.e. that the matrices, γ_{PQ} and ϕ_{jk} have been chosen as fixed sets of numbers independent of position and independent of the choice of gauge. For example, equation (4) fixes ϕ_{jk} as δ_{jk} and equation (5) shows that δ_{jk} is invariant under U(16) - transformations. Given an invariant inner product on each of the fundamental representations, one can construct an invariant inner product on any other representation (although it may not be unique). Since an invariant metric is just a collection of constants, it is not varied in finding field equations. Further, the covariant derivative of an invariant inner product is zero,

$$\nabla_a \phi_{jk} = 0 \ , \qquad \nabla_a \gamma_{PQ} = 0. \tag{23}$$

So the indices on fields may be raised and lowered before or after taking covariant derivatives. Conversely, if the connection satisfies equations (23) then it is always possible to reduce the group of both the bundle and the connection to the subgroup which leaves the metric invariant.

However, it is not always possible to find an invariant inner product. For example, the defining representation of GL(16,C) has no invariant hermitian form. In that case, in order to construct a scalar Lagrangian, it may be desirable to introduce a non-invariant inner product as a new dynamic variable. It would then be desirable to introduce a kinetic Lagrangian for the inner product. (Since the inner products on the fundamental representations induce inner products on the other representations, it is only necessary to introduce a kinetic Lagrangian for the inner products of the fundamental representations.)

For example, in a GL(16,C) gauge theory, the Lagrangian (22) is still appropriate for the leptons except that ϕ_{jk} is now a function of position in spacetime. The hermitian form, ϕ_{jk}, then induces a position dependent

group metric, γ_{PQ}, on the Lie algebra of GL(16,C) which can be used in the gauge Lagrangian (21). Since the covariant derivative of ϕ_{jk} may no longer vanish, one must be careful to specify whether indices are raised or lowered before or after taking covariant derivatives. Further, since $\nabla_a\phi_{jk}$ may not vanish, one can introduce a kinetic Lagrangian for ϕ_{jk} such as

$$\phi^{nj}(\nabla_a\phi_{jk})\phi^{km}(\nabla^a\phi_{mn}) \ , \tag{24}$$

where ϕ^{nj} is the inverse of ϕ_{jk}.

I point out that the inner product field, ϕ_{jk}, with the kinetic Lagrangian (24) behaves much like a Goldstone-Higgs field: By restricting to those gauges in which $\phi_{jk} = \delta_{jk}$, one reduces the group of the bundle from GL(16,C) to U(16). However, if $\nabla_a\phi_{jk} \neq 0$, then the GL(16,C) - connection does not reduce. Rather, it decomposes into a U(16) - connection and many residual gauge fields. The Lagrangian (24) becomes a mass term for all of the residual gauge fields with ϕ_{jk} absorbed as the longitudinal components. Thus the symmetry is broken. The inner product field, ϕ_{jk}, differs from a Goldstone-Higgs field in that there are no residual massive or massless scalar fields.

The inner product field, ϕ_{jk}, may also be regarded as a gauge potential in addition to the gauge connection, $A^P{}_a$. From this point of view, $\nabla_a\phi_{jk}$ should be regarded as the gauge field analogous to the gauge curvature, $F^P{}_{ab}$. When $F^P{}_{ab} = 0$, the dynamic field, $A^P{}_a$, may be eliminated from the theory. Similarly, when $\nabla_a\phi_{jk} = 0$, the dynamic field, ϕ_{jk}, may be eliminated. In spite of these analogies, in the following discussion of the matter Lagrangian, I treat ϕ_{jk} as just another source field included within ψ.

The gauge theory is called <u>global</u> if the connection, A^P_a, is required to be flat; i.e. the curvature vanishes, $F^P_{ab} = 0$. In that case it is always possible to find a choice of gauge in which the connection also vanishes, $A^P_a = 0$. Such a gauge is determined uniquely up to a global gauge transformation. The constraint, $F^P_{ab} = 0$, can be imposed by including in the Lagrangian a term which is a Lagrange multiplier times this constraint. Alternatively and equivalently, it can be imposed by setting $A^P_a = 0$ in the action, $S[\psi,A]$, and only varying ψ in the global gauge theory action,

$$S[\psi] = S[\psi,0] = \int L_M(\psi,\ldots,\partial^{(m)}\psi,0,\ldots,0)\ \sqrt{-g}\ d^4x. \qquad (25)$$

Any gauge theory which is not global is called <u>local</u>. Thus a local gauge theory may have some classical solutions in which $F^P_{ab} = 0$, but must also have some solutions in which $F^P_{ab} \neq 0$.

The matter Lagrangian (20) for a local gauge theory may be decomposed as follows: First, there is a constant term, L_C, which may be obtained by setting both $\psi = 0$ and $A = 0$:

$$L_C = L_M(0,\ldots,0). \qquad (26)$$

It is usually assumed that the energy density of the matter Lagrangian has a minimum and the constant term is usually adjusted so that the minimum is zero. Then the energy density is positive definite.

Second, there is a source Lagrangian, L_S, obtained by setting $A = 0$ in the non-constant part of the Lagrangian:

$$L_S(\psi,\ldots,\partial^{(m)}\psi) = L_M(\psi,\ldots,\partial^{(m)}\psi,0,\ldots,0) - L_C. \qquad (27)$$

Notice that $L_S + L_C$ is the Lagrangian of the corresponding global gauge theory.

Similarly, there is a gauge Lagrangian, L_A, obtained by setting $\psi = 0$ in the non-constant part of the Lagrangian:

$$L_A(A,\ldots,\partial^{(n)}A) = L_M(0,\ldots,0,A,\ldots,\partial^{(n)}A) - L_C. \tag{28}$$

The combination, $L_A + L_C$, is the Lagrangian for the source free gauge theory.

Finally, the remainder is the interaction Lagrangian, L_I, defined so that the total matter Lagrangian is

$$L_M = L_A(A,\ldots,\partial^{(n)}A) + L_S(\psi,\ldots,\partial^{(m)}\psi) + L_I(\psi,\ldots,\partial^{(m)}\psi,\ A,\ldots,\partial^{(n)}A) + L_C. \tag{29}$$

The sum $L_S + L_I$ is the interacting source Lagrangian, while $L_A + L_I$ is the interacting gauge Lagrangian.

The gauge theory is said to be <u>minimally coupled</u> if the interacting source Lagrangian can be obtained from the (global gauge theory) source Lagrangian by replacing all partial derivatives by covariant derivatives:

$$L_S(\psi,\ldots,\partial^{(m)}\psi) + L_I(\psi,\ldots,\partial^{(m)}\psi,\ A,\ldots,\partial^{(n)}A) = L_S(\psi,\ldots,\ \nabla^{(m)}\psi). \tag{30}$$

A gauge theory does not need to be minimally coupled although it is usually assumed. I will state the assumption when I make it.

I follow Fairchild [1977] and make a distinction between a gauge theory and a Yang-Mills theory. A gauge theory is called a <u>Yang-Mills theory</u> if the gauge Lagrangian is chosen as the Yang-Mills Lagrangian:

$$L_A = L_{YM} = -\frac{\hbar c}{16\pi g^2}\ F^P{}_{ab}\ F_P{}^{ab}\ , \tag{31}$$

where g is a dimensionless coupling constant. In the case of electromagnetism with the units chosen as in equation (18), the Yang-Mills Lagrangian (31) reduces to the Maxwell Lagrangian:

$$L_A = L_{Max} = -\frac{\hbar c}{16\pi\alpha} F^o{}_{ab} F_o{}^{ab}, \tag{32}$$

where $\alpha = e^2/(\hbar c)$ is the fine structure constant. A gauge theory does not need to be a Yang-Mills theory although it is usually assumed. I will state the assumption when I make it.

A weaker assumption is that the gauge Lagrangian is <u>minimally constructed</u>. This means that the gauge Lagrangian only depends on the gauge fields, $F^P{}_{ab}$, and a finite number of its covariant derivatives:

$$L_A = L_A(F,\ldots,\nabla^{(p)}F) \ . \tag{33}$$

In fact, at least for the case when L_A depends on no higher than first derivatives of $A^P{}_\mu$, the requirement that L_A be a scalar under gauge transformations implies that L_A is minimally constructed. The proof is similar to that for Noether's theorem. I do not know whether such a proof generalizes to higher derivatives.

2. Fibre Bundle Description of Metric-Connection Theories

Now that I have given a fibre bundle description of gauge theories, I give a corresponding description of metric-connection theories. The analogy will be obvious. Also the small but important differences will be clarified. As with the gauge theories, I discuss the fibre bundle description of metric-connection theories in more physical language. The spacetime symmetry group, G, can be any group which has a 4-dimensional real representation, R_T, acting on R^4. The most familiar groups are GL(4,R), O(3,1,R) = the Lorentz group, SL(2,C), and IO(3,1,R) = the inhomogeneous Lorentz or Poincare group. However, there are many other possible groups.

In the fibre bundle language, a metric-connection theory with spacetime symmetry group, G, prescribes the following geometrical objects:

a. a principal G-bundle, P, over spacetime M;

b. a G-vector bundle, E, associated to P with representation, R, acting on the typical fiber, V;

c. a global cross section, ψ, of the bundle, E;

d. a metric, g, on the tangent bundle, TM, to spacetime, M;

e. a soldering 1-form, θ, which makes TM into a G-vector bundle associated to P with representation, R_T; and

f. a connection 1-form, Γ, on the associated G-bundles, P, E and TM.

A particular metric-connection theory is specified by giving the spacetime symmetry group, G, the representations, R_T and R, and an action functional, $S[\psi,g,\theta,\Gamma]$. A classical solution of this theory is any choice of the manifold, M, and the associated G-bundles, P, E and TM, together with a cross section, ψ, a metric, g, a soldering 1-form, θ, and a connection, Γ, which make the action stationary.

In this description of metric-connection theories, I have assumed that there are no gauge fields for internal symmetries or that any internal gauge symmetry is global. To obtain a metric-connection theory with a local internal gauge symmetry, simply let G be the direct product of the spacetime symmetry group, G_1, and the internal symmetry group, G_2, (or some more complicated unification such as using GL(2,C) to unify gravity and electromagnetism as discussed in Section 3) and replace Γ by the direct sum of the spacetime connection, Γ, and the gauge connection, A. Then the above description of a metric-connection theory still holds, where R_T simply ignores the gauge part of G, and the action $S[\psi,g,\theta,\Gamma,A]$ is now also a function of the gauge connection.

The major difference between a metric-connection theory and any other gauge theory is that for a metric-connection theory, the tangent bundle, TM, to spacetime must be a G-vector bundle with a 4-dimensional real representation, R_T, and must be associated to the principal G-bundle, P. In other words, there must exist a preferred class of frames on TM, called the admissible tangent frames, and any two admissible frames at the same spacetime point must be related by a transformation belonging to the representation, R_T, of the group, G. For example, if G is the conformal orthogonal group, CO(3,1,R), then the admissible frames must be conformal orthonormal according to the metric, g. The set of all admissible frames at all points of M is called the admissible tangent frame bundle, $R_T(P)$, and forms a subbundle of the general linear frame bundle, GL(M), which consists of *all* frames at all points of M.

The bundle, $R_T(P)$, may be specified by giving a collection of local frame fields,

$$\overset{\alpha}{e}_{\mu} : U_\alpha \to TU_\alpha, \tag{1}$$

such that the domains cover M,

$$\bigcup_{\alpha} U_{\alpha} = M, \tag{2}$$

and on each overlap, $U_{\alpha} \cap U_{\beta}$, the frame fields are related by an $R_T(G)$ transformation,

$$\overset{\alpha}{e}_{\mu} = \overset{\beta}{e}_{\nu} [R_T \Lambda_{\alpha\beta}^{-1}]^{\nu}{}_{\mu} , \tag{3}$$

where the spacetime gauge transformation is

$$\Lambda_{\alpha\beta} : U_{\alpha} \cap U_{\beta} \to G. \tag{4}$$

The bundle, $R_T(P)$, is then the set of all frames related to the $\overset{\alpha}{e}_{\mu}$ by $R_T(G)$ transformations. Equivalently, the bundle, $R_T(P)$, may be specified by giving a collection of local 1-form frame fields,

$$\overset{\alpha}{\theta}{}^{\mu} : U_{\alpha} \to T^{*}U_{\alpha} , \tag{5}$$

(dual to the $\overset{\alpha}{e}_{\mu}$) such that the domains cover M and on each overlap, $U_{\alpha} \cap U_{\beta}$, they are related by

$$\overset{\alpha}{\theta}{}^{\mu} = [R_T \Lambda_{\alpha\beta}]^{\mu}{}_{\nu} \overset{\beta}{\theta}{}^{\nu} . \tag{6}$$

Such a collection of 1-forms, $\overset{\alpha}{\theta}{}^{\mu}$, is called a <u>base soldering 1-form</u>, θ, which makes TM into a G-vector bundle with representation R_T. (Notice the analogy between the definition of a soldering 1-form, θ, and the definition in Appendix B of a connection 1-form, Γ. Also notice that I now use $\Lambda_{\alpha\beta}$ as the gauge transformation instead of $g_{\alpha\beta}$ to avoid confusion with the metric, $g_{\mu\nu}$.)

For a given set of spacetime gauge transformations, $\Lambda_{\alpha\beta}$, as in (4), there may be more than one set of 1-form frame fields, $\overset{\alpha}{\theta}{}^{\mu}$, satisfying (6). Each set may specify a different way to identify TM as a G-vector bundle.

Thus the geometrical purpose of the soldring 1-form, θ, is to specify exactly how the admissible frame bundle, $R_T(P)$, sits inside of the general linear frame bundle, GL(M).

There may also exist a preferred class of spinor frames, called the admissible spinor frames, which are related by a representation, R_S, of the group, G. These make up the admissible spinor frame bundle, $R_S(P)$.

In Section 3, I discuss the possible spacetime symmetry groups, G, the corresponding principal bundle, P, the representation, R_T, the admissible tangent frames, and when appropriate, the representation, R_S, and the admissible spinor frames. I concentrate most on the homogeneous groups. A spacetime symmetry group, G, is called homogeneous if either R_T or R_S is effective. (A representation, R, of a group, G, is effective if for all $\Lambda \in G$ other than the identity, its representation, $R(\Lambda)$, is not the identity.) Otherwise, G is inhomogeneous. For example, the groups GL(4,R), O(3,1,R) = the Lorentz group, and SL(2,C) are homogeneous, while IO(3,1,R) = the inhomogeneous Lorentz or Poincare group is inhomogeneous. There are many other possible homogeneous groups, some of which are listed in Tables II.2 and II.4 at the end of Section 3. There are also inhomogeneous versions of all of those groups obtained as semidirect products of the homogeneous group with a translation group on which the homogeneous group acts. (See Section 3.)

In Section 4, I discuss the gravitational variables, g, θ and Γ. It is useful to point out that for some groups, certain variables disappear from the action. Thus, for O(3,1,R), SL(2,C), and their subgroups, the soldering form, θ, is a set of orthonormal frame fields. In these bases, the components of the metric, $g_{\mu\nu} = \eta_{\mu\nu}$, are constant. Hence, the metric components cannot be varied in the action. Similarly, for GL(4,R) the

soldering form, θ, becomes arbitrary and its variation in the action only produces an identity. (In fact, there are always the Noether identities relating the variations of g, θ and Γ. See Section III.5.) Furthermore, for some of the inhomogeneous groups, the soldering form, θ, gets absorbed into the connection, Γ.

As in the discussion of a gauge theory, the source fields are described by the cross section, ψ, of the bundle E. In a choice of spacetime gauge (say the α-gauge) the cross section, ψ, is described by a vector field, $\overset{\alpha}{\psi}$, whose components, $\overset{\alpha k}{\psi}$, k=1,..., N, are the components of all of the source fields. Under a change of the spacetime gauge the components mix according to the representation, R, of the group, G:

$$\overset{\alpha k}{\psi} = (R\Lambda_{\alpha\beta})^{k}{}_{j}\ \overset{\beta j}{\psi}. \tag{7}$$

Finally in Section 5, I study the action, $S[\psi,g,\theta,\Gamma,A]$. I first show that special relativity may be regarded as the global gauge theory of spacetime symmetries. Then I express the action as the integral of a Lagrangian and decompose the Lagrangian into a matter part, a gravitational part, and an interaction part. The matter Lagrangian may then be decomposed into a source part, a gauge part, and another interaction part as in Section 1. A discussion of minimal coupling is postponed to Section III.4. I mention the gravitational Lagrangian which I consider to be most analogous to the Yang-Mills Lagrangian but I postpone a detailed discussion of the gravitational Lagrangian to Chapter V.

3. The Principal G-Bundle: P

In this section I discuss the possible spacetime symmetry groups, G, and the corresponding principal G-bundle, P. In the process I discuss the representation, R_T, the admissible tangent frames, and the various tangent tensor bundles, $T^q_p M$. When appropriate, I discuss the admissible spinor frames and the various spinor tensor bundles, $T^{qs}_{pr} M$.

When the group, G, is homogeneous, the principal G-bundle, P, is usually realized as a frame bundle. As with a gauge theory, the class of admissible frames must be restricted so that the set of transformations between admissible frames coincides with the group, G.

First consider frames on the tanget bundle, TM. The set of all frames, e_μ, at all points of M is the general linear frame bundle, GL(M), which is a principal GL(4,R)-bundle. Using the metric, g, the set of all orthonormal frames is the orthonormal frame bundle, O(M,g), which is a principal O(3,1,R)-bundle. Similarly, using only the conformal metric, $\overset{*}{g}$, one obtains the bundle of conformal orthonormal frames, $CO(M,\overset{*}{g})$, which is a principal bundle with group, CO(3,1,R) = the conformal orthogonal or conformal Lorentz group. This bundle is particularly interesting since CO(3,1,R) is the largest subgroup of GL(4,R) which has "spinor representations." Since spinors are experimentally observed, I do not regard any subgroup of GL(4,R) larger than CO(3,1,R) as being physically relevant. However, I continue to consider such groups for completeness.

There are many other possible tangent frame bundles; some of which are listed in Table II.1 at the end of this section. Some of these are defined using the metric, g. Others use only the conformal metric, $\overset{*}{g}$, or the volume element, η_V. (The latter can be determined up to sign from the metric, g.) Each of these tangent frame bundles, P, is a principal bundle for some group,

G, which is a subgroup of GL(4,R). The appropriate groups are listed in Table II.1 and defined in Table II.2. In considering a metric-connection theory with spacetime symmetry group, $G \subset GL(4,R)$, the frames, e_μ, in the corresponding tangent frame bundle, P, will be called the admissible tangent frames.

A choice of spacetime gauge is a choice of admissible frame field, $\overset{\alpha}{e}_\mu$, for the tangent bundle, TM; i.e. a local cross section of the bundle, P. This induces a dual 1-form frame field, $\overset{\alpha}{\theta}{}^\mu$, on the cotangent bundle, T^*M, and induces a frame field, $\overset{\alpha}{e}_{\mu_1} \otimes \ldots \otimes \overset{\alpha}{e}_{\mu_p} \otimes \overset{\alpha\nu_1}{\theta} \otimes \ldots \otimes \overset{\alpha\nu_q}{\theta}$, on each of the tangent tensor bundles, T^q_pM. Under a change of admissible frame field,

$$\overset{\alpha}{e}_\mu = \overset{\beta}{e}_\nu (\Lambda^{-1}_{\alpha\beta})^\nu{}_\mu \,, \tag{1}$$

the components of a tangent vector,

$$X = \overset{\alpha\mu}{X} \overset{\alpha}{e}_\mu = \overset{\beta\nu}{X} \overset{\beta}{e}_\nu \in TM, \tag{2}$$

change according to

$$\overset{\alpha\mu}{X} = (\Lambda_{\alpha\beta})^\mu{}_\nu \overset{\beta\nu}{X} \,. \tag{3}$$

This defining representation of the group, G, denoted R^o_1, is the 4-dimensional real representation, R_T, referred to in the definition of a metric-connection theory. Thus,

$$R_T\Lambda_{\alpha\beta} = R^o_1\Lambda_{\alpha\beta} = \Lambda_{\alpha\beta} \,. \tag{4}$$

Similarly, under the frame transformation (1) a 1-form, $A \in T^*M$, transforms according to the representation, R^1_o, while a tensor, $B \in T^q_pM$, transforms according to the representation, R^q_p. Hence, P, TM, T^*M and all of the T^q_pM are associated G-bundles. These and other frame trans-

formation properties appear in Table II.7, while the infinitesimal versions appear in Table II.10. (Note that Table II.7 uses primed and unprimed indices to denote the choice of frame rather than the gauge labels, α and β.)

What about spinors? If one wants to consider spinors, then a subgroup, G, of GL(4,R) is no longer the correct spacetime symmetry group because the double valued spinor "representations" are not really representations. Rather, one must consider a spinor group, G, as the spacetime symmetry group. For symplicity, I only consider 2-component spinors, although a similar development could be done for 4-component spinors.

The fundamental spinor bundle, $T^{oo}_{\frac{1}{2}o}M$, is a 2-dimensional complex vector bundle over spacetime, M. Its dual bundle is the dual spinor bundle, $T^{\frac{1}{2}o}_{oo}M = (T^{oo}_{\frac{1}{2}o}M)^*$. Its conjugate bundle is the conjugate spinor bundle, $T^{oo}_{o\frac{1}{2}}M = \overline{T^{oo}_{\frac{1}{2}o}M}$,, and its dual conjugate bundle is the dual conjugate spinor bundle, $T^{o\frac{1}{2}}_{oo}M = \overline{(T^{oo}_{\frac{1}{2}o}M)^*}$. Taking tensor products of tensor powers of these bundles yields the spinor tensor bundles,

$$T^{qs}_{pr}M = (T^{oo}_{\frac{1}{2}o}M)^{2p} \otimes (T^{\frac{1}{2}o}_{oo}M)^{2q} \otimes (T^{oo}_{o\frac{1}{2}}M)^{2r} \otimes (T^{o\frac{1}{2}}_{oo}M)^{2s} , \qquad (5)$$

where p, q, r, and s are integral or half-integral.

(Note: If V is a finite dimensional complex vector space, then its dual space, V^*, is the set of complex linear functions on V; its conjugate space, $\bar{V}$, is the set of complex anti-linear functions on V^*; and its dual conjugate space, $\bar{V}^*$, is the set of complex anti-linear functions on V.

There is a canonical isomorphism, $(\bar{\ }) : V \to \bar{V} : \psi \to \bar{\psi}$, called <u>conjugation</u>, defined by $\bar{\psi}(\alpha) = \overline{\psi(\alpha)}$ for all $\alpha \in V^*$, where the last bar denotes complex conjugation. There is a similar conjugation, $(\bar{\ }) : V^* \to \bar{V}^*$. For bundles these definitions apply to fibres.)

Just as the tangent bundle, TM, has a metric, g, the spinor bundle, $T^{oo}_{\frac{1}{2}o}M$, must have an antisymmetric <u>spinor metric</u>, ε. This induces an <u>inverse metric</u>, ε^{-1}, on $T^{\frac{1}{2}o}_{oo}M$, a <u>conjugate metric</u>, $\bar{\varepsilon}$, on $T^{oo}_{o\frac{1}{2}}M$, and an <u>inverse conjugate metric</u>, $(\bar{\varepsilon})^{-1}$, on $T^{o\frac{1}{2}}_{oo}M$. Tensor products of ε, ε^{-1}, $\bar{\varepsilon}$, and $(\bar{\varepsilon})^{-1}$ provide induced metrics on the spinor tensor bundles, $T^{qs}_{pr}M$.

For $T^{oo}_{\frac{1}{2}o}M$ to be an acceptable choice for the fundamental spinor bundle (rather than just some arbitrary, 2-dimensional complex vector bundle with an antisymmetric metric) there must exist an isomorphism,

$$\sigma : T^{oo}_{\frac{1}{2}\frac{1}{2}}M_{Herm} \to TM \ , \tag{6}$$

from the Hermitian rank 2 spinor bundle to the tangent bundle. Further, the isomorphism, σ, must be an isometry from the metric, $s\,\varepsilon \otimes \bar{\varepsilon}$, to the metric, g. (Here, $s = \pm 1$ determines the signature of the metric, g, in that the Minkowski metric is $\eta = \mathrm{diag}(s,-s,-s,-s)$. Notice that the timelike convention, $s = +1$, makes $s\,\varepsilon \otimes \bar{\varepsilon}$, into the correct induced metric on $T^{oo}_{\frac{1}{2}\frac{1}{2}}M$ and so is nicer for dealing with spinors.)

The metric, ε, and the isomorphism, σ, can be written out more explicitly once bases have been specified.

Let u_A be a spinor basis for $T^{oo}_{\frac{1}{2}o}M$ at one point of M. Let v^A, $\bar{u}_{\dot{A}}$ and $\bar{v}^{\dot{A}}$ be the dual basis on $T^{\frac{1}{2}o}_{oo}M$, the conjugate basis on $T^{oo}_{o\frac{1}{2}}M$, and the dual conjugate basis on $T^{o\frac{1}{2}}_{oo}M$. These induce the basis,

$$u_{A_1} \otimes \ldots \otimes u_{A_p} \otimes v^{B_1} \otimes \ldots \otimes v^{B_q} \otimes \bar{u}_{\dot{A}_1} \otimes \ldots \otimes \bar{u}_{\dot{A}_r} \otimes \bar{v}^{\dot{B}_1} \otimes \ldots \otimes \bar{v}^{\dot{B}_s}, \tag{7}$$

on $T^{qs}_{pr}M$. In particular, $u_A \otimes \bar{u}_{\dot{A}}$ is the induced basis on $T^{oo}_{\frac{1}{2}\frac{1}{2}}M$. However, this basis is not Hermitian and so is not a basis for $T^{oo}_{\frac{1}{2}\frac{1}{2}}M_{\text{Herm}}$ as a 4-real dimensional vector bundle. A suitable basis for $T^{oo}_{\frac{1}{2}\frac{1}{2}}M_{\text{Herm}}$ is

$$\begin{aligned} &(u_o \otimes \bar{u}_{\dot{o}} + u_1 \otimes \bar{u}_{\dot{1}})/\sqrt{2}, \qquad (u_o \otimes \bar{u}_{\dot{1}} - u_1 \otimes \bar{u}_{\dot{o}})i/\sqrt{2}, \\ &(u_o \otimes \bar{u}_{\dot{1}} + u_1 \otimes \bar{u}_{\dot{o}})/\sqrt{2}, \qquad (u_o \otimes \bar{u}_{\dot{o}} - u_1 \otimes \bar{u}_{\dot{1}})/\sqrt{2}. \end{aligned} \tag{8}$$

The isomorphism, σ, is then used to define an induced basis for TM:

$$\begin{aligned} e_o &= \sigma(u_o \otimes \bar{u}_{\dot{o}} + u_1 \otimes \bar{u}_{\dot{1}})/\sqrt{2}, \\ e_1 &= \sigma(u_o \otimes \bar{u}_{\dot{1}} + u_1 \otimes \bar{u}_{\dot{o}})/\sqrt{2}, \\ e_2 &= \sigma(iu_o \otimes \bar{u}_{\dot{1}} - iu_1 \otimes \bar{u}_{\dot{o}})/\sqrt{2}, \\ e_3 &= \sigma(u_o \otimes \bar{u}_{\dot{o}} - u_1 \otimes \bar{u}_{\dot{1}})/\sqrt{2}. \end{aligned} \tag{9}$$

Letting θ^μ be the basis for T^*M dual to e_μ, the induced basis for T^q_pM is then

$$e_{\mu_1} \otimes \ldots \otimes e_{\mu_p} \otimes \theta^{\nu_1} \otimes \ldots \otimes \theta^{\nu_q}. \tag{10}$$

Notice that the real linear combinations of the basis (8) span $T^{oo}_{\frac{1}{2}\frac{1}{2}}M_{Herm}$, while the complex linear combinations span all of $T^{oo}_{\frac{1}{2}\frac{1}{2}}M$. Similarly, the real linear combinations of the basis (9) span TM, while the complex linear combinations span the complexified tangent bundle, $C \otimes TM$. Further, the complex linear combinations of the bases (10) span the complexified tangent tensor bundles, $C \otimes T^q_p M$.

Consequently, the isomorphism (6) extends by linearity to an isomorphism,

$$\sigma : T^{oo}_{\frac{1}{2}\frac{1}{2}}M \to C \otimes TM. \tag{11}$$

As a map between vector bundles, this isomorphism may be regarded as a tensor and expanded in the induced bases:

$$\sigma = e_\mu \otimes \sigma^\mu{}_{A\bar{A}}\, v^A \otimes \bar{v}^{\bar{A}} . \tag{12}$$

Using (9), one finds that the components of σ in the induced bases are just the Pauli matrices:

$$\sigma^0{}_{A\bar{A}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma^2{}_{A\bar{A}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$\sigma^1{}_{A\bar{A}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^3{}_{A\bar{A}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{13}$$

The isomorphism (11) may be inverted,

$$\sigma^{-1} : C \otimes TM \to T^{oo}_{\frac{1}{2}\frac{1}{2}}M , \tag{14}$$

and then restricted to the real subspace,

$$\sigma^{-1} : TM \to T^{oo}_{\frac{1}{2}\frac{1}{2}}M_{Herm} . \tag{15}$$

From (9),

$$\begin{aligned}
\sigma^{-1}(e_o) &= (u_o \otimes \bar{u}_{\bar{o}} + u_1 \otimes \bar{u}_{\bar{1}})/\sqrt{2} , \\
\sigma^{-1}(e_1) &= (u_o \otimes \bar{u}_{\bar{1}} + u_1 \otimes \bar{u}_{\bar{o}})/\sqrt{2} , \\
\sigma^{-1}(e_2) &= (u_o \otimes \bar{u}_{\bar{1}} - u_1 \otimes \bar{u}_{\bar{o}})i/\sqrt{2} , \\
\sigma^{-1}(e_3) &= (u_o \otimes \bar{u}_{\bar{o}} - u_1 \otimes \bar{u}_{\bar{1}})/\sqrt{2} ,
\end{aligned} \tag{16}$$

which is the basis (8) for $T^{oo}_{\frac{1}{2}\frac{1}{2}}M_{Herm}$. As a tensor, σ^{-1} may be expanded in the induced bases:

$$\sigma^{-1} = \theta^{\mu} \otimes (\sigma^{-1})_{\mu}{}^{A\bar{A}}\, u_A \otimes \bar{u}_{\bar{A}} . \tag{17}$$

Using (16), one finds that the components of σ^{-1} in the induced bases are

$$\begin{aligned}
(\sigma^{-1})_o{}^{A\bar{A}} &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \qquad (\sigma^{-1})_2{}^{A\bar{A}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} , \\
(\sigma^{-1})_1{}^{A\bar{A}} &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad (\sigma^{-1})_3{}^{A\bar{A}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .
\end{aligned} \tag{18}$$

Notice that $(\sigma^{-1})_2{}^{A\bar{A}}$ is *not* the Pauli matrix!

The same tensors, σ in (12) and σ^{-1} in (17), provide isomorphisms for the complexified cotangent bundle,

$$\sigma : C \otimes T^*M \to T^{\frac{1}{2}\frac{1}{2}}_{oo}M , \tag{19}$$

$$\sigma^{-1} : T^{\frac{1}{2}\frac{1}{2}}_{oo}M \to C \otimes T^*M \ , \tag{20}$$

which may be restricted to the real subspaces,

$$\sigma : T^*M \to T^{\frac{1}{2}\frac{1}{2}}_{oo}M_{Herm} \ , \tag{21}$$

$$\sigma^{-1} : T^{\frac{1}{2}\frac{1}{2}}_{oo}M_{Herm} \to T^*M \ . \tag{22}$$

Taking tensor products of tensor powers of σ and σ^{-1}, yields isomorphisms for the complexified tangent tensor bundles,

$$\sigma^{2p} \otimes (\sigma^{-1})^{2q} : T^{qq}_{pp}M \to C \otimes T^{2q}_{2p}M \ , \tag{23}$$

$$(\sigma^{-1})^{2p} \otimes \sigma^{2q} : C \otimes T^{2q}_{2p}M \to T^{qq}_{pp}M \ , \tag{24}$$

which may again be restricted to the real subspaces,

$$\sigma^{2p} \otimes (\sigma^{-1})^{2q} : T^{qq}_{pp}M_{Herm} \to T^{2q}_{2p}M \ , \tag{25}$$

$$(\sigma^{-1})^{2p} \otimes \sigma^{2q} : T^{2q}_{2p}M \to T^{qq}_{pp}M_{Herm} \ . \tag{26}$$

The spinor metric, ε, its inverse, ε^{-1}, its conjugate, $\bar{\varepsilon}$, and its inverse conjugate, $(\bar{\varepsilon})^{-1}$, may also be expanded in the bases u_A, v^A, $\bar{u}_{\bar{A}}$ and $\bar{v}^{\bar{A}}$:

$$\begin{aligned} \varepsilon &= \varepsilon_{AB}\, v^A \otimes v^B \ , & \varepsilon^{-1} &= \varepsilon^{AB}\, u_A \otimes u_B \ , \\ \bar{\varepsilon} &= \varepsilon_{\bar{A}\bar{B}}\, \bar{v}^{\bar{A}} \otimes \bar{v}^{\bar{B}} \ , & (\bar{\varepsilon})^{-1} &= \varepsilon^{\bar{A}\bar{B}}\, \bar{u}_{\bar{A}} \otimes \bar{u}_{\bar{B}} \ . \end{aligned} \tag{27}$$

That ε^{-1} is the inverse of ε and $(\bar{\varepsilon})^{-1}$ is the inverse of $\bar{\varepsilon}$, mean that

$$\varepsilon_{AB}\, \varepsilon^{CB} = \delta^C_A \ , \qquad \varepsilon_{\bar{A}\bar{B}}\, \varepsilon^{\bar{C}\bar{B}} = \delta^{\bar{C}}_{\bar{A}} \ . \tag{28}$$

That $\bar{\varepsilon}$ is the conjugate of ε and $(\bar{\varepsilon})^{-1}$ is the conjugate of ε^{-1}, mean that

$$\varepsilon_{\bar{A}\bar{B}} = \overline{\varepsilon_{AB}} \ , \qquad \varepsilon^{\bar{A}\bar{B}} = \overline{\varepsilon^{AB}} \ , \tag{29}$$

where the bar denotes complex conjugation.

The tensors ε_{AB}, ε^{AB}, $\varepsilon_{\bar{A}\bar{B}}$ and $\varepsilon^{\bar{A}\bar{B}}$ may be used to raise and lower spinor indices. However, since they are antisymmetric, a convention must be established. By convention, ε maps a spinor, $\psi = \psi^{A} u_{A} \in T^{oo}_{\frac{1}{2}o}M$, into the dual spinor, $\psi = \psi_{B} v^{B} \in T^{\frac{1}{2}o}_{oo}M$, while $\bar{\varepsilon}$ maps a conjugate spinor, $\chi = \chi^{\bar{A}} \bar{u}_{\bar{A}} \in T^{oo}_{o\frac{1}{2}}M$, into the dual conjugate spinor, $\chi = \chi_{\bar{B}} \bar{v}^{\bar{B}} \in T^{o\frac{1}{2}}_{oo}M$, where

$$\psi_{B} = \psi^{A} \varepsilon_{AB} \ , \qquad \chi_{\bar{B}} = \chi^{\bar{A}} \varepsilon_{\bar{A}\bar{B}} \ . \tag{30}$$

Consequently, using equation (28),

$$\chi^{C} = \varepsilon^{CB} \psi_{B} \ , \qquad \chi^{\bar{C}} = \varepsilon^{\bar{C}\bar{B}} \chi_{\bar{B}} \ . \tag{31}$$

Since equation (28) implies

$$\varepsilon^{CA} \varepsilon^{DB} \varepsilon_{AB} = \varepsilon^{CD} \ , \quad \varepsilon^{\bar{C}\bar{A}} \varepsilon^{\bar{D}\bar{B}} \varepsilon_{\bar{A}\bar{B}} = \varepsilon^{\bar{C}\bar{D}} \ , \tag{32}$$

these conventions (for inverse metrics and for raising and lowering indices) are consistent with saying that ε^{CD} and $\varepsilon^{\bar{C}\bar{D}}$ are just ε_{AB} and $\varepsilon_{\bar{A}\bar{B}}$ with their indices raised.

A spinor frame, u_{A}, is called <u>orthonormal</u> if in that frame the components of the spinor metric, ε, are

$$\varepsilon_{AB} = \varepsilon_{o} \ , \tag{33}$$

where I use the symbol, ε_{o}, to denote the fixed matrix,

$$\varepsilon_{o} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ . \tag{34}$$

Equations (28) and (29) then imply that for an orthonormal frame,

$$\varepsilon^{AB} = \varepsilon_{\bar{A}\bar{B}} = \varepsilon^{\bar{A}\bar{B}} = \varepsilon_o \ . \tag{35}$$

On the other hand, since ε must be antisymmetric, its components in a general spinor frame, u_A, are

$$\varepsilon_{AB} = \phi \ \varepsilon_o \ , \tag{36}$$

for some complex function, ϕ, called the spinor conformal factor. Then equations (28) and (29) imply

$$\varepsilon^{AB} = \phi^{-1} \ \varepsilon_o \ , \qquad \varepsilon_{\bar{A}\bar{B}} = \bar{\phi} \ \varepsilon_o \ , \qquad \varepsilon^{\bar{A}\bar{B}} = (\bar{\phi})^{-1} \ \varepsilon_o . \tag{37}$$

I use a general spinor frame except when I explicitly say it is orthonormal.

The requirement that the isomorphism, σ in (6), is an isometry from the metric, ${}^s\varepsilon \otimes \bar{\varepsilon}$, to the metric, g, implies that for all X, Y ε $T^{oo}_{\frac{1}{2}\frac{1}{2}}M$,

$$({}^s\varepsilon \otimes \bar{\varepsilon})(X, Y) = g(\sigma X, \sigma Y). \tag{38}$$

In indices this says

$${}^s\varepsilon_{AB} \ \varepsilon_{\bar{A}\bar{B}} \ X^{A\bar{A}} \ Y^{B\bar{B}} = g_{\mu\nu} \ \sigma^{\mu}{}_{A\bar{A}} \ X^{A\bar{A}} \ \sigma^{\nu}{}_{B\bar{B}} \ Y^{B\bar{B}} \ . \tag{39}$$

This must be true for all X and Y. Thus

$${}^s\varepsilon_{AB} \ \varepsilon_{\bar{A}\bar{B}} = g_{\mu\nu} \ \sigma^{\mu}{}_{A\bar{A}} \ \sigma^{\nu}{}_{B\bar{B}} \ , \tag{40}$$

or

$$g_{\mu\nu} = {}^s\varepsilon_{AB} \ \varepsilon_{\bar{A}\bar{B}} \ (\sigma^{-1})_{\mu}{}^{A\bar{A}} \ (\sigma^{-1})_{\nu}{}^{B\bar{B}} \ . \tag{41}$$

First notice that the equation (41) can be solved for the components of σ^{-1} in terms of the components of σ:

$$(\sigma^{-1})_{\mu}{}^{A\bar{A}} = s\, g_{\mu\nu}\, \varepsilon^{AB}\, \varepsilon^{\bar{A}\bar{B}}\, \sigma^{\nu}{}_{B\bar{B}} = s\, \sigma_{\mu}{}^{A\bar{A}} . \tag{42}$$

Thus, except for the factor of s, the components of σ^{-1} are just the components of σ with its indices raised and lowered using g, ε and $\bar{\varepsilon}$. (Again notice that the timelike convention, $s = +1$, is simpler for manipulating spinors.)

Second notice that equation (41) may be rewritten as the trace of a product of spinor matrices:

$$\begin{aligned} g_{\mu\nu} &= - s\, \varepsilon_{BA}\, (\sigma^{-1})_{\mu}{}^{A\bar{A}}\, \varepsilon_{\bar{A}\bar{B}}\, (\sigma^{-1})_{\nu}^{T\,\bar{B}B} \\ &= - s\, tr[\varepsilon\, (\sigma^{-1})_{\mu}\, \bar{\varepsilon}\, (\sigma^{-1})_{\nu}^{T}] . \end{aligned} \tag{43}$$

Explicitly computing the 10 independent components of $g_{\mu\nu}$, using equations (18), (36) and (37), one finds

$$g_{\mu\nu} = |\phi|^2\, \eta_{\mu\nu} = |\phi|^2\, \mathrm{diag}(s, -s, -s, -s). \tag{44}$$

Thus for an arbitrary spinor frame, u_A, the induced vector frame, e_μ, is always conformal orthonormal, and the vector conformal factor is $|\phi|^2$, the square of the absolute value of the spinor conformal factor. Further since the spinor frame, u_A, can be continuously rotated into any other spinor frame, u'_A, the induced vector frames, e_μ and e'_μ, have the same orientation and time orientation. By convention, these orientations are chosen as the standard ones.

Also notice from equation (44), that if u_A is orthonormal (ϕ=1), then e_μ is also orthonormal. In fact, even if ϕ is a pure phase ($|\phi|$=1), then e_μ is orthonormal. On the other hand, if u_A is a general spinor frame, then the vector metric, $g_{\mu\nu}$, determines the spinor metric, ε_{AB}, up to a

phase factor. (This phase factor can be used to incorporate electromagnetism into the theory of gravity as discussed below.)

This brings us back to the problem of classifying spinor frame bundles. The set of all spinor frames, u_A, at all points of M is the general linear spinor frame bundle, $GL_{Spin}(M)$, which is a principal GL(2,C)-bundle. The set of all orthonormal spinor frames is the orthonormal spinor frame bundle, $SL_{Spin}(M)$, which is a principal SL(2,C)-bundle. Between these extremes there are several other spinor frame bundles, P, listed in Table II.3. Each of these is a principal bundle for some group, $G \subset GL(2,C)$. The appropriate groups are listed in Table II.3 and defined in Table II.4.

As discussed above, each spinor frame, u_A induces an oriented, time oriented, conformal orthonormal, tangent frame, e_μ, on the tangent bundle, TM, according to equation (9). This defines a map

$$R_T : GL_{Spin}(M) \to CO_o(M). \tag{45}$$

Under this map, each spinor frame bundle, $P \subset GL_{Spin}(M)$, has an image, $R_T(P) \subset CO_o(M)$, which is a tangent frame bundle, and hence is a principle bundle for a group, $R_T(G) \subset CO_o(3,1,R)$, which forms a representation, R_T, of the group, $G \subset GL(2,C)$. The bundle $R_T(P)$ and group $R_T(G)$ are listed in Table II.3 and defined in Tables II.1 and II.2. The representation, R_T, is discussed in more detail below.

A choice of spacetime gauge is now a local cross section, $\overset{\alpha}{u}_A$, of the spinor frame bundle, P, or equivalently, a local frame field on the fundamental spinor bundle, $T^{OO}_{\frac{1}{2}O}M$. This induces a local cross section, $\overset{\alpha}{e}_\mu$, of the tangent frame bundle, $R_T(P)$, or equivalently, a local frame field on

the tangent bundle, TM. A change of spacetime gauge is a change of spinor frame field,

$$\overset{\alpha}{u}_A = \overset{\beta}{u}_B [U^{-1}_{\alpha\beta}]^B{}_A \ , \tag{46}$$

where $U_{\alpha\beta}$ belongs to the spinor group, $G \subset GL(2,C)$. This induces a change in the vector frame field,

$$\overset{\alpha}{e}_\mu = \overset{\beta}{e}_\nu [\Lambda^{-1}_{\alpha\beta}]^\nu{}_\mu \ , \tag{47}$$

where $\Lambda_{\alpha\beta} = R_T(U_{\alpha\beta})$ belongs to the tangent representation, $R_T(G) \subset CO_o(3,1,R)$.

Under the spinor frame transformation (46), a spinor, $\psi = \overset{\alpha A}{\psi}\ \overset{\alpha}{u}_A = \overset{\beta B}{\psi}\ \overset{\beta}{u}_B$, transforms according to

$$\overset{\alpha A}{\psi} = [U_{\alpha\beta}]^A{}_B\ \overset{\beta B}{\psi} \ , \tag{48}$$

and a tangent vector, $X = \overset{\alpha\mu}{X}\ \overset{\alpha}{e}_\mu = \overset{\beta\nu}{X}\ \overset{\beta}{e}_\nu$, transforms according to

$$\overset{\alpha}{X} = [\Lambda_{\alpha\beta}]^\mu{}_\nu\ \overset{\beta\nu}{X}. \tag{49}$$

Similar transformations are induced on all of the spinor tensor bundles, $T^{qs}_{pr}M$, and the tangent tensor bundles, T^q_pM. Thus, P, $R_T(P)$, $T^{qs}_{pr}M$, and T^q_pM are all associated G-bundles. These and other transformation properties under a change of spinor frame, appear in Table II.8, while the infinitesimal versions appear in Table II.11. (Note that Table II.8 uses primed and unprimed indices to denote the choice of frame rather than the gauge labels, α and β.)

How is the matrix, $\Lambda_{\alpha\beta} = R_T(U_{\alpha\beta})$, related to $U_{\alpha\beta}$; i.e. what is the representation, R_T? From equations (17) and (46),

$$\begin{aligned}\sigma^{-1}(\overset{\alpha}{e}_\mu) &= (\sigma^{-1})_\mu{}^{A\bar{A}}\ \overset{\alpha}{u}_A \otimes \overset{\alpha}{\bar{u}}_{\bar{A}} \\ &= (\sigma^{-1})_\mu{}^{A\bar{A}}\ \overset{\beta}{u}_B \otimes \overset{\beta}{\bar{u}}_{\bar{B}}\ [U^{-1}_{\alpha\beta}]^B{}_A\ [\bar{U}^{-1}_{\alpha\beta}]^{\bar{B}}{}_{\bar{A}} \ .\end{aligned} \tag{50}$$

On the other hand, from equations (47) and (17),

$$\sigma^{-1}(\overset{\alpha}{e}_{\mu}) = \sigma^{-1}(\overset{\beta}{e}_{\nu})\ [\Lambda^{-1}_{\alpha\beta}]^{\nu}{}_{\mu}$$

$$= (\sigma^{-1})_{\nu}{}^{B\bar{B}}\ \overset{\beta}{u}_{B} \otimes \overset{\beta}{\bar{u}}_{\bar{B}}\ [\Lambda^{-1}_{\alpha\beta}]^{\nu}{}_{\mu}\ . \tag{51}$$

Equating coefficients leads to

$$[\Lambda^{-1}_{\alpha\beta}]^{\nu}{}_{\mu} = \sigma^{\nu}{}_{B\bar{B}}\ (\sigma^{-1})_{\mu}{}^{A\bar{A}}\ [U^{-1}_{\alpha\beta}]^{B}{}_{A}\ [\bar{U}^{-1}_{\alpha\beta}]^{\bar{B}}{}_{\bar{A}}\ , \tag{52}$$

$$[\Lambda_{\alpha\beta}]^{\mu}{}_{\nu} = \sigma^{\mu}{}_{A\bar{A}}\ (\sigma^{-1})_{\nu}{}^{B\bar{B}}\ [U_{\alpha\beta}]^{A}{}_{B}\ [\bar{U}_{\alpha\beta}]^{\bar{A}}{}_{\bar{B}}\ . \tag{53}$$

Equation (53) defines the representation,

$$R_T : GL(2,C) \to CO_o(3,1,R), \tag{54}$$

which may be restricted to any group, $G \subset GL(2,C)$. Table II.4 defines several such subgroups, G, lists the image, $R_T(G)$, and also the kernel of the restriction, $R_T|_G$; i.e. the set of all $U \in G$ such that $R_T(U) = \mathbb{1} \in R_T(G)$.

Several spinor groups deserve special attention. The restriction of R_T to SL(2,C),

$$R_T : SL(2,C) \to O_o(3,1,R), \tag{55}$$

is the usual 2-1 representation of SL(2,C) onto the restricted Lorentz group. It is 2-1 because its kernel, $\{1,-1\} \subset SL(2,C)$, has 2 elements. SL(2,C) is an appropriate spacetime symmetry group for discussing spinors in a metric-connection theory with a Cartan connection.

Similarly, the 2-1 covering of the restricted conformal Lorentz group is

$$R_T : CL(2,C) \to CO_o(3,1,R)\ , \tag{56}$$

where CL(2,C) is the subgroup of GL(2,C) for which the determinant is real and positive. CL(2,C) is the spacetime symmetry group for the spinor frame bundle, $CL_{Spin}(M)$, which requires the spinor conformal factor, ϕ, to be real and positive. It is an appropriate group for discussing spinors in a theory with a Weyl-Cartan connection.

If one were to consider a theory in which spinors are coupled to both an electromagnetic potential and a Cartan connection, one would usually consider the group, SL(2,C) × U(1). However, one could instead use the group, PL(2,C), which is the subgroup of GL(2,C) for which the determinant is a pure phase. These groups have the same Lie algebra and so require the same connection fields. However, their multiplet structure could differ because they are related by a 2-1 homomorphism,

$$SL(2,C) \times U(1) \to PL(2,C) : (U, e^{i\theta}) \to e^{i\theta} U. \qquad (57)$$

The spacetime symmetry group, PL(2,C), corresponds to the spinor frame bundle, $PL_{Spin}(M)$, for which the spinor conformal factor, ϕ, is a pure phase ($|\phi|=1$). The kernel of the tangent representation,

$$R_T : PL(2,C) \to O_o(3,1,R), \qquad (58)$$

is now $U(1) = S^1 = \{e^{i\theta} \mathbb{1}\}$. Thus in addition to gauging the Lorentz group, the group, PL(2,C), also gauges the phase factors of the spinor wave functions. I believe this is in fact what is usually done.

Similarly, one could go all the way and use the full spinor group, GL(2,C), to describe spinors in a Weyl-Cartan theory with electromagnetism.

At the beginning of this section, I said that the principal bundle, P, is usually taken as a frame bundle, and then I proceeded to discuss the tangent and spinor frame bundles. I now briefly mention two generalizations in which P is not simply a frame bundle.

First, recall that for any matrix $U \in GL(2,C)$, its tangent representation, $\Lambda = R_T(U)$, as defined by equation (53), must belong to the *restricted* conformal orthogonal group, $CO_o(3,1,R)$. Thus none of the groups, $G = GL(2,C)$, appropriate to the spinor frame bundles in Table II.3, can be used to describe time reversal, T, space reversal, P, or space-time reversal, $\mathsf{PT} = -\mathbb{1}$, of the vector frames. One way to include one or all of these discrete operations is to extend the group, G, to the direct product of G with one of the finite groups, $\{\mathbb{1}, \mathsf{T}\}$, $\{\mathbb{1}\ \mathsf{P}\}$, $\{\mathbb{1}, -\mathbb{1}\}$, or $\{\mathbb{1}, -\mathbb{1}, \mathsf{T}, \mathsf{P}\}$. This is done in Table II.4. A principal bundle for the extended spinor group can then be constructed by taking the union of 2 or 4 copies of the original principal bundle for the unextended group.

Second, all of the spacetime symmetry groups discussed so far (those in Tables II.2 and II.4) have been homogeneous groups, However, as pointed out in Section 2, there are also inhomogeneous groups. For each homogeneous group (now called H) there is an inhomogeneous group, G, which is the semi-direct product of H with the 4-dimensional real translation group, T(4,R), where the semi-direct product is implemented by the representation, R_T, of H acting on $T(4,R) = R^4$. Thus

$$G = H \times_{R_T} T(4,R) = \{(\Lambda,a) : \Lambda\varepsilon H,\ a \in T(4,R)\}, \tag{59}$$

with the product rule

$$(\Lambda,a) \circ (M,b) = (\Lambda M,\ (R_T\Lambda)b + a). \tag{60}$$

The representation, R_T, is then extended to G by ignoring the translations:

$$R_T(\Lambda,a) = R_T\Lambda\ . \tag{61}$$

Given the principal H-bundle, Q, a principal G-bundle, P, can be constructed as the intrinsic Cartesian product,

$$P = Q \times TM. \tag{62}$$

(By an intrinsic Cartesian product I mean that each fibre of P is the Cartesian product of the corresponding fibres of Q and TM.) The bundle, P, is called the affine version of the bundle, Q. (See Kobayashi and Nomizu [1963] pp. 125-130 for a more detailed discussion of affine frame bundles.)

For each of the tangent groups, H, listed in Table II.2, the inhomogeneous version, G, is denoted by prefixing an "I" to the symbol for H and the word "inhomogeneous" to its name. Correspondingly, for each of the tangent frame bundles, Q, listed in Table II.1, the affine version, P, is denoted by prefixing an "A" to the symbol and appending "affine" to the name. Thus, for example, the oriented, time oriented, conformal orthogonal, affine, tangent frame bundle,

$$ACO_o(M,\overset{*}{g}) = CO_o(M,\overset{*}{g}) \times TM, \tag{63}$$

is a principal bundle for the inhomogeneous, restricted, conformal orthogonal group,

$$ICO_o(3,1,R) = CO_o(3,1,R) \times_{R_T} T(4,R). \tag{64}$$

On the other hand, the affine spinor group, G, obtained (by the semidirect product with T(4,R) via R_T) from a spinor group, H, listed in Table II.4, is denoted by prefixing an "A" to the symbol and "affine" to the name. The corresponding affine spinor frame bundle is also denoted by prefixing "A" to the symbol and "affine" to the name. Thus, for example, the orthonormal, affine, spinor frame bundle,

$$ASL_{Spin}(M) = SL_{Spin}(M) \times TM\ , \tag{65}$$

is a principal bundle for the affine, special linear group,

$$ASL(2,C) = SL(2,C) \times_{R_T} T(4,R) \ . \tag{66}$$

I emphasize that the affine spinor groups are denoted with "A" and "affine" rather than "I" and "inhomogeneous." The latter would refer to the semi-direct product,

$$G = H \times_{R_S} T(2,C) \ , \tag{67}$$

of H with the 2-dimensional complex translation group, T(2,C), via the defining spinor representation, R_S. The corresponding principal bundle is the intrinsic Cartesian product,

$$P = Q \times T^{OO}_{\frac{1}{2}O}M, \tag{68}$$

which would be denoted by prefixing an "I" to the symbol for Q and "inhomogeneous" to its name. Thus, for example, the orthonormal, inhomogeneous, spinor frame bundle,

$$ISL_{Spin}(M) = SL_{Spin}(M) \times T^{OO}_{\frac{1}{2}O}M, \tag{69}$$

is a principal bundle for the inhomogeneous, special linear group,

$$ISL(2,C) = SL(2,C) \times_{R_S} T(2,C).$$

I conclude this section by commenting that there are probably many other ways to construct spacetime symmetry groups and their principal bundles which I have not included here.

TABLE II.1 TANGENT FRAME BUNDLES

BUNDLE P	CLASS OF FRAMES e_μ	GROUP G
GL(M)	general linear (all)	GL(4,R)
$GL_o(M)$	oriented	$GL_o(4,R)$
$VL(M,\eta_V)$	unit volume	VL(4,R)
$SL(M,\eta_V)$	oriented, unit volume	SL(4,R)
$CO(M,\overset{*}{g})$	conformal orthonormal	CO(3,1,R)
$CO_o(M,\overset{*}{g})$	oriented, time oriented, CO	$CO_o(3,1,R)$
$CO_+(M,\overset{*}{g})$	oriented, CO	$CO_+(3,1,R)$
$CO_T(M,\overset{*}{g})$	time oriented, CO	$CO_T(3,1,R)$
$CO_S(M,\overset{*}{g})$	space oriented, CO	$CO_S(3,1,R)$

(continued)

TABLE II.1 continued

BUNDLE P	CLASS OF FRAMES e_μ	GROUP G
$O(M,g)$	orthonormal	$O(3,1,R)$
$O_o(M,g)$	oriented, time oriented, 0	$O_o(3,1,R)$
$SO(M,g)$	oriented, 0	$SO(3,1,R)$
$O_T(M,g)$	time oriented, 0	$O_T(3,1,R)$
$O_S(M,g)$	space oriented, 0	$O_S(3,1,R)$

NOTATION AND TERMINOLOGY:

M = spacetime manifold

g = metric field

$\overset{*}{g}$ = conformal metric field

η_V = volume 4-form field

$\eta_{\mu\nu}$ = Minkowski metric

e_μ = frame field

t = local time function

orthonormal: $g(e_\mu,e_\nu) = \eta_{\mu\nu}$

conformal orthonormal: $g(e_\mu,e_\nu) = |\det g|^{\frac{1}{4}} \eta_{\mu\nu}$

unit volume: $\det g(e_\mu,e_\nu) = -1$ or $\eta_V(e_o,e_1,e_2,e_3) = \pm 1$

oriented: $\eta_V(e_o,e_1,e_2,e_3) > 0$

time oriented: e_o future directed or $e_o(t) > 0$

space oriented: $\eta_V(e_o,e_1,e_2,e_3) \cdot e_o(t) > 0$

TABLE II.2 GAUGE GROUPS FOR TANGENT FRAME BUNDLES

GROUP G	DEFINITION	CONNECTED COMPONENTS	NAME
$GL(4,R)$	$= \{\Lambda \in M(4,R) : \det \Lambda \neq 0\}$	$= GL_o(4,R) \cup T \cdot GL_o(4,R)$	= general linear
$GL_o(4,R)$	$= \{\Lambda \in GL(4,R) : \det \Lambda > 0\}$	= conn comp of 1 in $GL(4,R)$	= restricted (or proper) GL
$VL(4,R)$	$= \{\Lambda \in GL(4,R) : \det \Lambda = \pm 1\}$	$= SL(4,R) \cup T \cdot SL(4,R)$	= volumetric linear
$SL(4,R)$	$= \{\Lambda \in GL(4,R) : \det \Lambda = 1\}$	= conn comp of 1 in $VL(4,R)$	= special linear or unimodular
$CO(3,1,R)$	$= \{\Lambda \in GL(4,R) : \Lambda^T \eta \Lambda = \lvert\det\Lambda\rvert^{\frac{1}{2}} \eta\}$	$= CO_o(3,1,R) \cup (-1) \cdot CO_o(3,1,R) \cup T \cdot CO_o(3,1,R) \cup P \cdot CO_o(3,1,R)$	= conformal orthogonal or conformal Lorentz
$CO_o(3,1,R)$	$= \{\Lambda \in CO(3,1,R) : \det \Lambda > 0, \Lambda^o{}_o > 0\}$	= conn comp of 1 in $CO(3,1,R)$	= restricted CO
$CO_+(3,1,R)$	$= \{\Lambda \in CO(3,1,R) : \det \Lambda > 0\}$	$= CO_o(3,1,R) \cup (-1) \cdot CO_o(3,1,R)$	= proper CO
$CO_T(3,1,R)$	$= \{\Lambda \in CO(3,1,R) : \Lambda^o{}_o > 0\}$	$= CO_o(3,1,R) \cup P \cdot CO_o(3,1,R)$	= orthochronous CO
$CO_S(3,1,R)$	$= \{\Lambda \in CO(3,1,R) : \Lambda^o{}_o \det\Lambda > 0\}$	$= CO_o(3,1,R) \cup T \cdot CO_o(3,1,R)$	= orthochorous CO

(continued)

TABLE II.2 continued

GROUP G	DEFINITION	CONNECTED COMPONENTS	NAME
$O(3,1,R)$	$= \{\Lambda \in GL(4,R) : \Lambda^T \eta \Lambda = \eta\}$	$= O_o(3,1,R) \cup (-\mathbb{1}) \cdot O_o(3,1,R) \cup T \cdot O_o(3,1,R) \cup P \cdot O_o(3,1,R)$	= orthogonal or Lorentz
$O_o(3,1,R)$	$= \{\Lambda \in O(3,1,R) : \det\Lambda = 1, \Lambda^o{}_o \geq 1\}$	= conn comp of $\mathbb{1}$ in $O(3,1,R)$	= restricted O
$SO(3,1,R)$	$= \{\Lambda \in O(3,1,R) : \det\Lambda = 1\}$	$= O_o(3,1,R) \cup (-\mathbb{1}) \cdot O_o(3,1,R)$	= proper (or special) O
$O_T(3,1,R)$	$= \{\Lambda \in O(3,1,R) : \Lambda^o{}_o \geq 1\}$	$= O_o(3,1,R) \cup P \cdot O_o(3,1,R)$	= orthochronous O
$O_S(3,1,R)$	$= \{\Lambda \in O(3,1,R) : \Lambda^o{}_o \det\Lambda \geq 1\}$	$= O_o(3,1,R) \cup T \cdot O_o(3,1,R)$	= orthochorous O

NOTATION AND TERMINOLOGY:

general matrix = $\Lambda = (\Lambda^\mu{}_\nu) \in M(4,R)$

identity = $\mathbb{1} = (\delta^\mu{}_\nu) = \text{diag}(1,1,1,1)$

spacetime reversal = $-\mathbb{1} = (-\delta^\mu{}_\nu) = \text{diag}(-1,-1,-1,-1)$

time reversal = $T = (T^\mu{}_\nu) = \text{diag}(-1,1,1,1)$

space reversal = $P = (P^\mu{}_\nu) = \text{diag}(1,-1,-1,-1)$

Minkowski metric = $\eta = (\eta^\mu{}_\nu) = \text{diag}(s,-s,-s,-s)$

$s = \pm 1$

volumetric = volume preserving

conformal orthogonal = angle preserving

orthogonal = length preserving

restricted = connected to $\mathbb{1}$

proper = orientation preserving

special = unimodular = unit determinant

orthochronous = time orientation preserving

orthochorous = space orientation preserving

TABLE II.3 SPINOR FRAME BUNDLES

BUNDLE P	CLASS OF SPINOR FRAMES u_A	GROUP G	R^4 - REPRESENTATION $R_T(G)$	$R_T(P)$	CLASS OF TANGENT FRAMES e_μ
$GL_{Spin}(M)$	general linear (all)	GL(2,C)	$CO_o(3,1,R)$	$CO_o(M,\overset{*}{g})$	oriented, time oriented conformal orthonormal
$RL_{Spin}(M)$	real conformal orthonormal $\varepsilon = \pm\lvert\det\varepsilon\rvert^{\frac{1}{2}}\varepsilon_o$ or $\det\varepsilon > 0$	RL(2,C)	$CO_o(3,1,R)$	$CO_o(M,\overset{*}{g})$	oriented, time oriented conformal orthonormal
$CL_{Spin}(M)$	positive real conf. orthon. $\varepsilon = \lvert\det\varepsilon\rvert^{\frac{1}{2}}\varepsilon_o$	CL(2,C)	$CO_o(3,1,R)$	$CO_o(M,\overset{*}{g})$	oriented, time oriented conformal orthonormal
$PL_{Spin}(M)$	phase conformal orthonormal $\varepsilon = a\,\varepsilon_o$ with $\lvert a\rvert = 1$ or $\lvert\det\varepsilon\rvert = 1$	PL(2,C)	$O_o(3,1,R)$	$O_o(M,g)$	oriented, time oriented orthonormal
$VL_{Spin}(M)$	unit volume $\varepsilon = \pm\,\varepsilon_o$ or $\det\varepsilon = 1$	VL(2,C)	$O_o(3,1,R)$	$O_o(M,g)$	oriented, time oriented orthonormal
$SL_{Spin}(M)$	orthonormal $\varepsilon = \varepsilon_o$	SL(2,C)	$O_o(3,1,R)$	$O_o(M,g)$	oriented, time oriented orthonormal

$$\varepsilon_o = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

TABLE II.4 GAUGE GROUPS FOR SPINOR FRAME BUNDLES

GROUP G	DEFINITION	NAME	R^4 - REPRESENTATION KERNEL	$R_T(G)$
$GL(2,C)$	$= \{ U \in M(2,C) : \det U \neq 0 \}$	= general linear	S^1	$CO_\circ(3,1,R)$
$GL'(2,C)$	$= GL \times \{ 1, -1, T, P \}$	= fully extended GL	S^1	$CO(3,1,R)$
$GL_+(2,C)$	$= GL \times \{ 1, -1 \}$	= proper extended GL	S^1	$CO_+(3,1,R)$
$GL_T(2,C)$	$= GL \times \{ 1, P \}$	= orthochronous extended GL	S^1	$CO_T(3,1,R)$
$GL_S(2,C)$	$= GL \times \{ 1, T \}$	= orthochorous extended GL	S^1	$CO_S(3,1,R)$
$RL(2,C)$	$= \{ U \in GL(2,C) : \det U \in R \}$	= real conformal linear	$\{ 1, -1, i, -i \}$	$CO_\circ(3,1,R)$
$RL'(2,C)$	$= RL \times \{ 1, -1, T, P \}$	= fully extended RL	$\{ 1, -1, i, -i \}$	$CO(3,1,R)$
$RL_+(2,C)$	$= RL \times \{ 1, -1 \}$	= proper extended RL	$\{ 1, -1, i, -i \}$	$CO_+(3,1,R)$
$RL_T(2,C)$	$= RL \times \{ 1, P \}$	= orthochronous extended RL	$\{ 1, -1, i, -i \}$	$CO_T(3,1,R)$
$RL_S(2,C)$	$= RL \times \{ 1, T \}$	= orthochorous extended RL	$\{ 1, -1, i, -i \}$	$CO_S(3,1,R)$

(continued)

TABLE II.4 continued

GROUP G	DEFINITION	NAME	R^4 - REPRESENTATION KERNEL	$R_T(G)$
$CL(2,C)$	$= \{ U \in RL(2,C) : \det U > 0 \}$	= positive real conf. linear	{ 1, -1 }	$CO_o(3,1,R)$
$CL'(2,C)$	$= CL \times \{ 1, -1, T, P \}$	= fully extended CL	{ 1, -1 }	$CO(3,1,R)$
$CL_+(2,C)$	$= CL \times \{ 1, -1 \}$	= proper extended CL	{ 1, -1 }	$CO_+(3,1,R)$
$CL_T(2,C)$	$= CL \times \{ 1, P \}$	= orthochronous extended CL	{ 1, -1 }	$CO_T(3,1,R)$
$CL_S(2,C)$	$= CL \times \{ 1, T \}$	= orthochorous extended CL	{ 1, -1 }	$CO_S(3,1,R)$
$PL(2,C)$	$= \{ U \in GL(2,C) : \lvert \det U \rvert = 1 \}$	= phase conformal linear	S^1	$O_o(3,1,R)$
$PL'(2,C)$	$= PL \times \{ 1, -1, T, P \}$	= fully extended PL	S^1	$O(3,1,R)$
$PL_+(2,C)$	$= PL \times \{ 1, -1 \}$	= proper extended PL	S^1	$SO(3,1,R)$
$PL_T(2,C)$	$= PL \times \{ 1, P \}$	= orthochronous extended PL	S^1	$O_T(3,1,R)$
$PL_S(2,C)$	$= PL \times \{ 1, T \}$	= orthochorous extended PL	S^1	$O_S(3,1,R)$

(continued)

TABLE II.4 continued

GROUP G	DEFINITION	NAME	R^4 - REPRESENTATION KERNEL	$R_T(G)$
$VL(2,C)$	$= \{ U \in GL(2,C) : \det U = \pm 1 \}$	volumetric linear	{ 1, -1, i, -i }	$O_o(3,1,R)$
$VL'(2,C)$	$= VL \times \{ 1, -1, T, P \}$	fully extended VL	{ 1, -1, i, -i }	$O(3,1,R)$
$VL_+(2,C)$	$= VL \times \{ 1, -1 \}$	proper extended VL	{ 1, -1, i, -i }	$SO(3,1,R)$
$VL_T(2,C)$	$= VL \times \{ 1, P \}$	orthochronous extended VL	{ 1, -1, i, -i }	$O_T(3,1,R)$
$VL_S(2,C)$	$= VL \times \{ 1, T \}$	orthochorous extended VL	{ 1, -1, i, -i }	$O_S(3,1,R)$
$SL(2,C)$	$= \{ U \in GL(2,C) : \det U = 1 \}$	special linear	{ 1, -1 }	$O_o(3,1,R)$
$SL'(2,C)$	$= SL \times \{ 1, -1, T, P \}$	fully extended SL	{ 1, -1 }	$O(3,1,R)$
$SL_+(2,C)$	$= SL \times \{ 1, -1 \}$	proper extended SL	{ 1, -1 }	$SO(3,1,R)$
$SL_T(2,C)$	$= SL \times \{ 1, P \}$	orthochronous extended SL	{ 1, -1 }	$O_T(3,1,R)$
$SL_S(2,C)$	$= SL \times \{ 1, T \}$	orthochorous extended SL	{ 1, -1 }	$O_S(3,1,R)$

TABLE II.5 LIE ALGEBRAS FOR TANGENT AND SPINOR FRAME BUNDLES

LIE ALGEBRA $\mathcal{L}G$	DEFINITION	GROUPS G	GENERATORS	CONNECTION RESTRICTIONS
$\mathcal{L}GL(4,R)$	$= M(4,R)$ $=$ {4×4 real matrices}	$GL_o(4,R)$ $GL(4,R)$	$\{E^{\beta}{}_{\alpha} : \alpha,\beta = 0,1,2,3\}$ or $\{\sigma_{\beta\alpha}, D_o, \tau_{\beta\alpha}, D_k :$ $0 \leq \beta < \alpha \leq 3,\ k = 1,2,3\}$	$\nabla_\mu g_{\alpha\beta}$ arbitrary
$\mathcal{L}SL(4,R)$	$= \{X \in M(4,R) :$ $X^{\mu}{}_{\mu} = 0\}$	$SL(4,R)$ $VL(4,R)$	$\{\sigma_{\beta\alpha}, \tau_{\beta\alpha}, D_k :$ $0 \leq \beta < \alpha \leq 3,\ k = 1,2,3\}$	$g^{\alpha\beta}\ \nabla_\mu g_{\alpha\beta} = 0$
$\mathcal{L}GL(2,C)$	$= M(2,C)$ $=$ {2×2 complex matrices}	$GL(2,C)$ $GL'(2,C)$ $GL_+(2,C)$ $GL_T(2,C)$ $GL_S(2,C)$	$\{\sigma_{\beta\alpha}, D_o, D_I : 0 \leq \beta < \alpha \leq 3\}$	$\nabla_\mu \epsilon_{AB}$ arbitrary

(continued)

TABLE II.5 continued

LIE ALGEBRA $\mathcal{L}_G$	DEFINITION	GROUPS G	GENERATORS	CONNECTION RESTRICTIONS
$\mathcal{L}PL(2,C)$	$= \{X \in M(2,C) : Re\ X^A{}_A = 0\}$	$PL(2,C)$ $PL'(2,C)$ $PL_+(2,C)$ $PL_T(2,C)$ $PL_S(2,C)$	$\{\sigma_{\beta\alpha}, D_I : 0 \leq \beta < \alpha \leq 3\}$	$Re\ \epsilon^{AB} \nabla_\mu \epsilon_{AB} = 0$
$\mathcal{L}CO(3,1,R)$	$= \{X \in M(4,R) : \eta_{\mu(\alpha} X^\mu{}_{\beta)} = \frac{1}{4} \eta_{\alpha\beta} X^\mu{}_\mu\}$	$CO_\circ(3,1,R)$ $CO(3,1,R)$ $CO_+(3,1,R)$ $CO_T(3,1,R)$ $CO_S(3,1,R)$	$\{\sigma_{\beta\alpha}, D_\circ : 0 \leq \beta < \alpha \leq 3\}$	$\nabla_\mu g_{\alpha\beta} = -\frac{1}{2} \lambda_\mu\ g_{\alpha\beta}$
$= \mathcal{L}CL(2,C)$	$= \{X \in M(2,C) : Im\ X^A{}_A = 0\}$	$CL(2,C)$, $RL(2,C)$ $CL'(2,C)$, $RL'(2,C)$ $CL_+(2,C)$, $RL_+(2,C)$ $CL_T(2,C)$, $RL_T(2,C)$ $CL_S(2,C)$, $RL_S(2,C)$		$Im\ \epsilon^{AB} \nabla_\mu \epsilon_{AB} = 0$

(continued)

TABLE II.5 continued

LIE ALGEBRA $\mathcal{L}G$	DEFINITION	GROUPS G	GENERATORS	CONNECTION RESTRICTIONS
$\mathcal{L}O(3,1,R)$	$= \{X \in M(4,R) : \eta_{\mu(\alpha} X^{\mu}{}_{\beta)} = 0\}$	$O_o(3,1,R)$ $O(3,1,R)$ $SO(3,1,R)$ $O_T(3,1,R)$ $O_S(3,1,R)$	$\{\sigma_{\beta\alpha} : 0 \leq \beta < \alpha \leq 3\}$	$\nabla_\mu g_{\alpha\beta} = 0$
$= \mathcal{L}SL(2,C)$	$= \{X \in M(2,C) : X^{A}{}_{A} = 0\}$	$SL(2,C)$, $VL(2,C)$ $SL'(2,C)$, $VL'(2,C)$ $SL_+(2,C)$, $VL_+(2,C)$ $SL_T(2,C)$, $VL_T(2,C)$ $SL_S(2,C)$, $VL_S(2,C)$		$\nabla_\mu \varepsilon_{AB} = 0$

The vector matrices, $E^{\beta}{}_{\alpha} \in M(4,R)$, have matrix components, $(E^{\beta}{}_{\alpha})^{\mu}{}_{\nu} = \delta^{\mu}_{\alpha}\,\delta^{\nu}_{\beta}$. From these one obtains, $E_{\beta\alpha} = \eta_{\beta\gamma}\,E^{\gamma}{}_{\alpha}$, $\sigma_{\beta\alpha} = E_{\beta\alpha} - E_{\alpha\beta}$, $\tau_{\beta\alpha} = E_{\beta\alpha} + E_{\alpha\beta}$, $D_k = E^{k}{}_{k} - E^{o}{}_{o}$ (k not summed), and $D_o = 2\,E^{\alpha}{}_{\alpha} = 2\,1_4$.

The spinor matrices, $\sigma_{\beta\alpha}$, D_o, $D_I \in M(2,C)$, have matrix components,

$$(\sigma_{12})^{A}{}_{B} = -\,i\,(\sigma_{03})^{A}{}_{B} = s\,\frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (\sigma_{13})^{A}{}_{B} = i\,(\sigma_{02})^{A}{}_{B} = -\,s\,\frac{i}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$(\sigma_{23})^{A}{}_{B} = -\,i\,(\sigma_{01})^{A}{}_{B} = s\,\frac{i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad (D_o)^{A}{}_{B} = -\,i\,(D_I)^{A}{}_{B} = (1_2)^{A}{}_{B} = \delta^{A}{}_{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

TABLE II.6 COORDINATE TRANSFORMATIONS

1. coordinates	$x'^{a'} = x'^{a'}(x^b)$
2. coordinate vector basis	$\partial_{a'} = \frac{\partial x^b}{\partial x'^{a'}} \partial_b$
3. coordinate 1-form basis	$dx'^{a'} = \frac{\partial x'^{a'}}{\partial x^b} dx^b$
4. scalar	$f'(x') = f(x)$
5. coordinate components of	
a. tangent tensor	$\psi^{(a')}(x') = R^q_p(\frac{\partial x'}{\partial x})^{(a')}{}_{(b)} \psi^{(b)}(x)$

where

$$\psi^{(a)} = \psi^{a_1 \cdots a_p}{}_{a_{p+1} \cdots a_{p+q}}$$

$$R^q_p(\frac{\partial x'}{\partial x})^{(a')}{}_{(b)} = \frac{\partial x'^{a'_1}}{\partial x^{b_1}} \cdots \frac{\partial x'^{a'_p}}{\partial x^{b_p}} \frac{\partial x^{b_{p+1}}}{\partial x'^{a'_{p+1}}} \cdots \frac{\partial x^{b_{p+q}}}{\partial x'^{a'_{p+q}}}$$

(continued)

TABLE II.6 continued

b. metric

$$g_{a'b'}(x') = \frac{\partial x^c}{\partial x'^{a'}} \frac{\partial x^d}{\partial x'^{b'}} g_{cd}(x)$$

c. connection coefficients

$$\Gamma^{a'}{}_{b'c'} = \frac{\partial x^f}{\partial x'^{c'}} \frac{\partial x^e}{\partial x'^{b'}} \left(\frac{\partial x'^{a'}}{\partial x^d} \Gamma^d{}_{ef} - \frac{\partial^2 x'^{a'}}{\partial x^f \partial x^e} \right)$$

where

$$\nabla_{\partial_c} \partial_b = \Gamma^a{}_{bc}\, \partial_a$$

6. frame components of

a. tangent tensor

$$\psi'^{(\alpha)}(x') = \psi^{(\alpha)}(x)$$

where

$$\psi^{(\alpha)} = \psi^{\alpha_1 \cdots \alpha_p}{}_{\alpha_{p+1} \cdots \alpha_{p+q}}$$

b. metric

$$g'_{\alpha\beta}(x') = g_{\alpha\beta}(x)$$

c. connection coefficients

$$\Gamma'^{\alpha}{}_{\beta\gamma}(x') = \Gamma^{\alpha}{}_{\beta\gamma}(x)$$

where

$$\nabla_{e_\gamma} e_\beta = \Gamma^\alpha{}_{\beta\gamma}\, e_\alpha$$

(continued)

TABLE II.6 continued

7. mixed components of

a. vector frame $$e_{\alpha}{}^{a'}(x') = \frac{\partial x'^{a'}}{\partial x^{b}} e_{\alpha}{}^{b}(x)$$

b. 1-form frame $$\theta^{\alpha}{}_{a'}(x') = \frac{\partial x^{b}}{\partial x'^{a'}} \theta^{\alpha}{}_{b}(x)$$

c. soldering isomorphism $$\sigma_{\alpha}{}^{a'}(x') = \frac{\partial x'^{a'}}{\partial x^{b}} \sigma_{\alpha}{}^{b}(x)$$

d. inverse soldering isomorphism $$(\sigma^{-1})^{\alpha}{}_{a'}(x') = \frac{\partial x^{b}}{\partial x'^{a'}} (\sigma^{-1})^{\alpha}{}_{b}(x)$$

e. connection coefficients $$\Gamma^{\alpha}{}_{\beta a'}(x') = \frac{\partial x^{b}}{\partial x'^{a'}} \Gamma^{\alpha}{}_{\beta b}(x)$$

where

$$\nabla_{\partial_a} e_{\beta} = \Gamma^{\alpha}{}_{\beta a} e_{\alpha}$$

(continued)

TABLE II.6 continued

8. spinor frame components of

a. spinor tensor $\psi'^{(A)}(x') = \psi^{(A)}(x)$

where

$$\psi^{(A)} = \psi^{A_1 \cdots A_p}{}_{A_{p+1} \cdots A_{p+q}}{}^{\bar{A}_1 \cdots \bar{A}_r}{}_{\bar{A}_{r+1} \cdots \bar{A}_{r+s}}$$

b. spinor metric $\varepsilon'_{AB}(x') = \varepsilon_{AB}(x)$

c. spinor connection coefficients $\Gamma^{A}{}_{Ba'}(x') = \frac{\partial x^b}{\partial x'^{a'}} \Gamma^{A}{}_{Bb}(x)$

where

$$\nabla_{\partial_a} u_B = \Gamma^{A}{}_{Ba} u_A$$

TABLE II.7 TANGENT FRAME TRANSFORMATIONS

1. vector frame $\quad e'_{\alpha'} = (\Lambda^{-1})^{\beta}{}_{\alpha'}\, e_{\beta}$

2. 1-form frame $\quad \theta'^{\alpha'} = \Lambda^{\alpha'}{}_{\beta}\, \theta^{\beta}$

3. scalar $\quad f$ invariant

4. frame components of

   a. tangent tensor $\quad \psi^{(\alpha')} = R^{q}_{p}(\Lambda)^{(\alpha')}{}_{(\beta)}\, \psi^{(\beta)}$

      where

$$\psi^{(\alpha)} = \psi^{\alpha_1 \cdots \alpha_p}{}_{\alpha_{p+1} \cdots \alpha_{p+q}}$$

$$R^{q}_{p}(\Lambda)^{(\alpha')}{}_{(\beta)} = \Lambda^{\alpha'_1}{}_{\beta_1} \cdots \Lambda^{\alpha'_p}{}_{\beta_p} (\Lambda^{-1})^{\beta_{p+1}}{}_{\alpha'_{p+1}} \cdots (\Lambda^{-1})^{\beta_{p+q}}{}_{\alpha'_{p+q}}$$

(continued)

TABLE II.7 continued

b. metric

$$g_{\alpha'\beta'} = (\Lambda^{-1})^{\gamma}{}_{\alpha'}\,(\Lambda^{-1})^{\delta}{}_{\beta'}\,g_{\gamma\delta}$$

SPECIAL CASES:

O(3,1,R):

$$g_{\alpha'\beta'} = g_{\alpha\beta} = \eta_{\alpha\beta} \quad \text{invariant}$$

CO(3,1,R):

$$g_{\alpha'\beta'} = \Omega'\,\eta_{\alpha'\beta'} \qquad g_{\alpha\beta} = \Omega\,\eta_{\alpha\beta} \qquad \Omega' = (\det \Lambda)^{-2}\,\Omega$$

c. connection coefficients

$$\Gamma^{\alpha'}{}_{\beta'\gamma'} = (\Lambda^{-1})^{\varepsilon}{}_{\gamma'}\,\Lambda^{\alpha'}{}_{\delta}\left[\,\Gamma^{\delta}{}_{\rho\varepsilon}\,(\Lambda^{-1})^{\rho}{}_{\beta'} + e_{\varepsilon}(\Lambda^{-1})^{\delta}{}_{\beta'}\,\right]$$

where

$$\nabla_{e_\gamma} e_\beta = \Gamma^{\alpha}{}_{\beta\gamma}\, e_\alpha$$

5. coordinate components of

a. tangent tensor

$$\psi^{(a)} = \psi^{a_1\cdots a_p}{}_{a_{p+1}\cdots a_{p+q}} \quad \text{invariant}$$

b. metric

$$g_{ab} \quad \text{invariant}$$

c. connection coefficients

$$\Gamma^{a}{}_{bc} \quad \text{invariant}$$

(continued)

TABLE II.7 continued

6. mixed components of

a. vector frame $e'_{\alpha'}{}^{a} = (\Lambda^{-1})^{\beta}{}_{\alpha'} e_{\beta}{}^{a}$

b. 1-form frame $\theta'^{\alpha'}{}_{a} = \Lambda^{\alpha'}{}_{\beta} \theta^{\beta}{}_{a}$

c. soldering isomorphism $\sigma'_{\alpha'}{}^{a} = (\Lambda^{-1})^{\beta}{}_{\alpha'} \sigma_{\beta}{}^{a}$

d. inverse soldering isomorphism $(\sigma'^{-1})^{\alpha'}{}_{a} = \Lambda^{\alpha'}{}_{\beta} (\sigma^{-1})^{\beta}{}_{a}$

e. connection coefficients $\Gamma^{\alpha'}{}_{\beta'a} = \Lambda^{\alpha'}{}_{\gamma} \left[\Gamma^{\gamma}{}_{\delta a} (\Lambda^{-1})^{\delta}{}_{\beta'} + \partial_a (\Lambda^{-1})^{\gamma}{}_{\beta'} \right]$

where

$$\nabla_{\partial_a} e_\beta = \Gamma^{\alpha}{}_{\beta a} e_\alpha$$

TABLE II.8 SPINOR FRAME TRANSFORMATIONS

1. spinor frame

$$u'_{A'} = (U^{-1})^{B}{}_{A'}\, u_B$$

2. dual spinor frame

$$v'^{A'} = U^{A'}{}_{B}\, v^B$$

3. conjugate spinor frame

$$\bar{u}'_{\bar{A}'} = (\overline{U^{-1}})^{\bar{B}}{}_{\bar{A}'}\, \bar{u}_{\bar{B}}$$

4. dual conjugate spinor frame

$$\bar{v}'^{\bar{A}'} = \bar{U}^{\bar{A}'}{}_{\bar{B}}\, \bar{v}^{\bar{B}}$$

5. spinor frame components of

   a. spinor tensor

$$\psi^{(A')} = R^{qs}_{pr}(U)^{(A')}{}_{(B)}\, \psi^{(B)}$$

where

$$\psi^{(A)} = \psi^{A_1\cdots A_p}{}_{A_{p+1}\cdots A_{p+q}}{}^{\bar{A}_1\cdots\bar{A}_r}{}_{\bar{A}_{r+1}\cdots\bar{A}_{r+s}}$$

$$R^{qs}_{pr}(U)^{(A')}{}_{(B)} = U^{A'_1}{}_{B_1}\cdots U^{A'_p}{}_{B_p}(U^{-1})^{B_{p+1}}{}_{A'_{p+1}}\cdots(U^{-1})^{B_{p+q}}{}_{A'_{p+q}}\,\bar{U}^{\bar{A}'_1}{}_{\bar{B}_1}\cdots\bar{U}^{\bar{A}'_r}{}_{\bar{B}_r}(\overline{U^{-1}})^{\bar{B}_{r+1}}{}_{\bar{A}'_{r+1}}\cdots(\overline{U^{-1}})^{\bar{B}_{r+s}}{}_{\bar{A}'_{r+s}}$$

(continued)

TABLE II.8 continued

b. spinor metric $\varepsilon_{A'B'} = (U^{-1})^{C}{}_{A'}\,(U^{-1})^{D}{}_{B'}\,\varepsilon_{CD}$

c. spinor connection coefficients $\Gamma^{A'}{}_{B'a} = U^{A'}{}_{C}\left[\Gamma^{C}{}_{Da}\,(U^{-1})^{D}{}_{B'} + \partial_a (U^{-1})^{C}{}_{B'}\right]$

where

$$\nabla_{\partial_a} u_B = \Gamma^{A}{}_{Ba}\, u_A$$

A spinor frame transformation induces a tangent frame transformation (between oriented, time oriented, conformal orthonormal frames) by the conformal Lorentz transformation:

$$\Lambda^{\alpha'}{}_{\beta} = \sigma^{\alpha'}{}_{A'\bar{A}'}\,(\sigma^{-1})_{\beta}{}^{B\bar{B}}\,U^{A'}{}_{B}\,\bar{U}^{\bar{A}'}{}_{\bar{B}}$$

The induced transformation properties are those listed in Table II.7 plus

6. Pauli spin matrices are invariant: $\sigma^{\alpha'}{}_{A'\bar{A}'} = \Lambda^{\alpha'}{}_{\beta}\,\sigma^{\beta}{}_{B\bar{B}}\,(U^{-1})^{B}{}_{A'}\,\overline{(U^{-1})}^{\bar{B}}{}_{\bar{A}'} = \sigma^{\alpha}{}_{A\bar{A}}$

TABLE II.9 COORDINATE VARIATIONS

DEFINITION: An infinitesimal coordinate transformation has the form,

$$x'^a = x^a + \varepsilon^a(x^b) \quad , \qquad x^b = x'^b - \varepsilon^b(x'^a) \quad ,$$

so that,

$$\frac{\partial x'^a}{\partial x^b} = \delta^a_b + \partial_b \varepsilon^a \quad , \qquad \frac{\partial x^b}{\partial x'^a} = \delta^b_a - \partial_a \varepsilon^b \quad .$$

The coordinate variation of any geometrical object, Φ , is defined as,

$$\delta\Phi = \Phi'(x) - \Phi(x) = \Phi'(x') - \Phi(x) - \varepsilon^b \partial_b \Phi \quad ,$$

to first order in $\varepsilon^a(x)$. Notice that the coordinate variation commutes with coordinate partial derivatives,

$$\delta\partial_a\Phi = [\ \partial'_a\Phi'(x')\ ]_{x'\to x} - \partial_a\Phi(x) = \partial_a\Phi'(x) - \partial_a\Phi(x) = \partial_a\delta\Phi \quad ,$$

but not with frame partial derivatives,

$$\delta e_\alpha\Phi = e_\alpha{}^a\, \delta\partial_a\Phi + (\delta e_\alpha{}^a)\, \partial_a\Phi = e_\alpha\delta\Phi + (\, -\varepsilon^b \partial_b e_\alpha{}^a + e_\alpha{}^b \partial_b \varepsilon^a\,)\, \partial_a\Phi \quad .$$

(continued)

TABLE II.9 continued

1. coordinates $\delta x^a = \epsilon^a$

2. scalar $\delta f = - \epsilon^b \partial_b f$

3. coordinate components of

a. tangent tensor $\delta \psi^{(a)} = - \epsilon^b \partial_b \psi^{(a)} + (\partial_d \epsilon^c) \, R^q_p (E^d{}_c)^{(a)}{}_{(b)} \, \psi^{(b)}$

where

$$\psi^{(a)} = \psi^{a_1 \cdots a_p}{}_{a_{p+1} \cdots a_{p+q}}$$

$$R^q_p (E^d{}_c)^{(a)}{}_{(b)} = \sum_{k=1}^{p} \left(\delta^{a_1}_{b_1} \cdots \delta^{a_k}_{c} \, \delta^{d}_{b_k} \cdots \delta^{b_{p+q}}_{a_{p+q}} \right) - \sum_{k=1}^{q} \left(\delta^{a_1}_{b_1} \cdots \delta^{b_{p+k}}_{c} \, \delta^{d}_{a_{p+k}} \cdots \delta^{b_{p+q}}_{a_{p+q}} \right)$$

b. metric $\delta g_{cd} = - \epsilon^b \partial_b g_{cd} - (\partial_c \epsilon^b) \, g_{bd} - (\partial_d \epsilon^b) \, g_{cb}$

c. connection coefficients $\delta \Gamma^c{}_{da} = - \epsilon^b \partial_b \Gamma^c{}_{da} + (\partial_b \epsilon^c) \, \Gamma^b{}_{da} - (\partial_d \epsilon^b) \, \Gamma^c{}_{ba} - (\partial_a \epsilon^b) \, \Gamma^c{}_{db} - \partial_d \partial_a \epsilon^c$

where

$$\nabla_{\partial_a} \partial_d = \Gamma^c{}_{da} \, \partial_c$$

(continued)

TABLE 11.9 continued

4. frame components of

a. tangent tensor $\qquad \delta\psi^{(\alpha)} = - \varepsilon^{b} \partial_{b} \psi^{(\alpha)}$

where

$$\psi^{(\alpha)} = \psi^{\alpha_1 \cdots \alpha_p}{}_{\alpha_{p+1} \cdots \alpha_{p+q}}$$

b. metric $\qquad \delta g_{\alpha\beta} = - \varepsilon^{b} \partial_{b} g_{\alpha\beta}$

SPECIAL CASES:

O(3,1,R): $\quad g_{\alpha\beta} = \eta_{\alpha\beta} \qquad \delta g_{\alpha\beta} = 0$

CO(3,1,R): $\quad g_{\alpha\beta} = \Omega\, \eta_{\alpha\beta} \qquad \delta g_{\alpha\beta} = (\delta\Omega)\, \eta_{\alpha\beta} = \Omega^{-1} (\delta\Omega)\, g_{\alpha\beta} \qquad \delta\Omega = - \varepsilon^{b} \partial_{b} \Omega$

c. connection coefficients $\qquad \delta \Gamma^{\alpha}{}_{\beta\gamma} = - \varepsilon^{b} \partial_{b} \Gamma^{\alpha}{}_{\beta\gamma}$

where

$$\nabla_{e_\gamma} e_\beta = \Gamma^{\alpha}{}_{\beta\gamma}\, e_\alpha$$

(continued)

TABLE II.9 continued

5. mixed components of

a. vector frame $\quad \delta e_\alpha{}^a = - \varepsilon^b \partial_b e_\alpha{}^a + (\partial_b \varepsilon^a)\, e_\alpha{}^b$

b. 1-form frame $\quad \delta \theta^\alpha{}_a = - \varepsilon^b \partial_b \theta^\alpha{}_a - (\partial_a \varepsilon^b)\, \theta^\alpha{}_b$

c. soldering isomorphism $\quad \delta \sigma_\alpha{}^a = - \varepsilon^b \partial_b \sigma_\alpha{}^a + (\partial_b \varepsilon^a)\, \sigma_\alpha{}^b$

d. inverse soldering isomorphism $\quad \delta (\sigma^{-1})^\alpha{}_a = - \varepsilon^b \partial_b (\sigma^{-1})^\alpha{}_a - (\partial_a \varepsilon^b)\, (\sigma^{-1})^\alpha{}_b$

e. connection coefficients $\quad \delta \Gamma^\alpha{}_{\beta a} = - \varepsilon^b \partial_b \Gamma^\alpha{}_{\beta a} - (\partial_a \varepsilon^b)\, \Gamma^\alpha{}_{\beta b}$

where

$$\nabla_{\partial_a} e_\beta = \Gamma^\alpha{}_{\beta a}\, e_\alpha$$

(continued)

TABLE II.9 continued

6. spinor frame components of

a. spinor tensor $\qquad \delta\psi^{(A)} = - \varepsilon^{b} \partial_{b} \psi^{(A)}$

where

$$\psi^{(A)} = \psi^{A_1 \cdots A_p \quad \bar{A}_1 \cdots \bar{A}_r}_{A_{p+1} \cdots A_{p+q} \quad \bar{A}_{r+1} \cdots \bar{A}_{r+s}}$$

b. spinor metric $\qquad \delta\varepsilon_{AB} = - \varepsilon^{b} \partial_{b} \varepsilon_{AB}$

SPECIAL CASES:

SL(2,C): $\qquad \varepsilon_{AB} = \varepsilon_{\circ AB} \qquad \delta\varepsilon_{AB} = 0$

GL(2,C): $\qquad \varepsilon_{AB} = \phi\, \varepsilon_{\circ AB} \qquad \delta\varepsilon_{AB} = (\delta\phi)\, \varepsilon_{\circ AB} = \phi^{-1} (\delta\phi)\, \varepsilon_{AB} \qquad \delta\phi = - \varepsilon^{b} \partial_{b} \phi$

c. spinor connection coefficients $\qquad \delta\Gamma^{A}_{\ Ba} = - \varepsilon^{b} \partial_{b} \Gamma^{A}_{\ ABa} - (\partial_{a} \varepsilon^{b})\, \Gamma^{A}_{\ Bb}$

where

$$\nabla_{\partial_a} u_B = \Gamma^{A}_{\ Ba}\, u_A$$

TABLE II.10 TANGENT FRAME VARIATIONS

DEFINITION: An infinitesimal tangent frame transformation has the form,

$$\Lambda^{\alpha}{}_{\beta} = \delta^{\alpha}_{\beta} + \lambda^{\alpha}{}_{\beta} \quad , \qquad (\Lambda^{-1})^{\beta}{}_{\alpha} = \delta^{\beta}_{\alpha} - \lambda^{\beta}{}_{\alpha} \quad .$$

The tangent frame variation of a geometrical object, Φ , is defined as,

$$\delta\Phi = \Phi' - \Phi \quad ,$$

to first order in $\lambda^{\alpha}{}_{\beta}$. Notice that the tangent frame variation commutes with coordinate partial derivatives,

$$\delta\partial_a\Phi = \partial_a\Phi' - \partial_a\Phi = \partial_a\delta\Phi \quad ,$$

but not with frame partial derivatives,

$$\delta e_\alpha\Phi = e_\alpha{}^a\, \delta\partial_a\Phi + (\delta e_\alpha{}^a)\, \partial_a\Phi = e_\alpha\delta\Phi - \lambda^{\beta}{}_{\alpha}\, e_\beta\Phi \quad .$$

1.	vector frame	$\delta e_\alpha = -\lambda^{\beta}{}_{\alpha}\, e_\beta$
2.	1-form frame	$\delta\theta^{\alpha} = \lambda^{\alpha}{}_{\beta}\, \theta^{\beta}$
3.	scalar	$\delta f = 0$

(continued)

TABLE II.10 continued

4. frame components of

a. tangent tensor $\qquad \delta\psi^{(\alpha)} = R^{q}_{p}(\lambda)^{(\alpha)}{}_{(\beta)}\ \psi^{(\beta)}$

where

$$\psi^{(\alpha)} = \psi^{\alpha_1 \cdots \alpha_p}{}_{\alpha_{p+1} \cdots \alpha_{p+q}}$$

$$R^{q}_{p}(\lambda)^{(\alpha)}{}_{(\beta)} = \sum_{k=1}^{p} \left(\delta^{\alpha_1}_{\beta_1} \cdots \lambda^{\alpha_k}{}_{\beta_k} \cdots \delta^{\beta_{p+q}}_{\alpha_{p+q}} \right) - \sum_{k=1}^{q} \left(\delta^{\alpha_1}_{\beta_1} \cdots \lambda^{\beta_{p+k}}{}_{\alpha_{p+k}} \cdots \delta^{\beta_{p+q}}_{\alpha_{p+q}} \right)$$

b. metric $\qquad \delta g_{\alpha\beta} = - \lambda^{\gamma}{}_{\alpha}\ g_{\gamma\beta} - \lambda^{\gamma}{}_{\beta}\ g_{\alpha\gamma} = - 2\ \lambda_{(\alpha\beta)}$

SPECIAL CASES:

O(3,1,R): $\quad g_{\alpha\beta} = \eta_{\alpha\beta} \qquad \delta g_{\alpha\beta} = - 2\ \lambda_{(\alpha\beta)} = 0$

CO(3,1,R): $\quad g_{\alpha\beta} = \Omega\ \eta_{\alpha\beta} \qquad \delta g_{\alpha\beta} = (\delta\Omega)\ \eta_{\alpha\beta} = \Omega^{-1}\ (\delta\Omega)\ g_{\alpha\beta} = - 2\ \lambda_{(\alpha\beta)} = - \tfrac{1}{2}\ \lambda^{\gamma}{}_{\gamma}\ g_{\alpha\beta}$

c. connection coefficients $\qquad \delta\Gamma^{\alpha}{}_{\beta\gamma} = \lambda^{\alpha}{}_{\delta}\ \Gamma^{\delta}{}_{\beta\gamma} - \lambda^{\delta}{}_{\beta}\ \Gamma^{\alpha}{}_{\delta\gamma} - \lambda^{\delta}{}_{\gamma}\ \Gamma^{\alpha}{}_{\beta\delta} - e_{\gamma}\lambda^{\alpha}{}_{\beta}$

where

$$\nabla_{e_\gamma} e_\beta \quad \Gamma^{\alpha}{}_{\beta\gamma}\ e_\alpha$$

(continued)

TABLE II.10 continued

5. coordinate components of

a. tangent tensor	$\delta\psi^{(a)} = \delta\psi^{a_1\cdots a_p}{}_{a_{p+1}\cdots a_{p+q}} = 0$
b. metric	$\delta g_{ab} = 0$
c. connection coefficients	$\delta\Gamma^{a}{}_{bc} = 0$

6. mixed components of

a. vector frame	$\delta e_{\alpha}{}^{a} = -\lambda^{\beta}{}_{\alpha}\, e_{\beta}{}^{a}$
b. 1-form frame	$\delta\theta^{\alpha}{}_{a} = \lambda^{\alpha}{}_{\beta}\, \theta^{\beta}{}_{a}$
c. soldering isomorphism	$\delta\sigma_{\alpha}{}^{a} = -\lambda^{\beta}{}_{\alpha}\, \sigma_{\beta}{}^{a}$
d. inverse soldering isomorphism	$\delta(\sigma^{-1})^{\alpha}{}_{a} = \lambda^{\alpha}{}_{\beta}\, (\sigma^{-1})^{\beta}{}_{a}$
e. connection coefficients	$\delta\Gamma^{\alpha}{}_{\beta a} = \lambda^{\alpha}{}_{\gamma}\,\Gamma^{\gamma}{}_{\beta a} - \lambda^{\gamma}{}_{\beta}\,\Gamma^{\alpha}{}_{\gamma a} - \partial_a\lambda^{\alpha}{}_{\beta} = -\nabla_a\lambda^{\alpha}{}_{\beta}$

where

$$\nabla_{\partial_a} e_\beta = \Gamma^{\alpha}{}_{\beta a}\, e_\alpha$$

TABLE II.11 SPINOR FRAME VARIATIONS

DEFINITION: An infinitesimal spinor frame transformation has the form,

$$U^{A}{}_{B} = \delta^{A}_{B} + u^{A}{}_{B} \quad , \qquad (U^{-1})^{B}{}_{A} = \delta^{B}_{A} - u^{B}{}_{A} \quad .$$

This induces an infinitesimal tangent frame transformation with

$$\lambda^{\alpha}{}_{\beta} = \sigma^{\alpha}{}_{A\bar{A}} \, (\sigma^{-1})_{\beta}{}^{B\bar{B}} \, (\, u^{A}{}_{B} \, \delta^{\bar{A}}_{\bar{B}} + \delta^{A}_{B} \, \bar{u}^{\bar{A}}{}_{\bar{B}} \,) \quad .$$

The spinor frame variation of a geometrical object, Φ , is defined as,

$$\delta\Phi = \Phi' - \Phi \quad ,$$

to first order in $u^{A}{}_{B}$. The tangent frame variations listed in Table II.10 may be converted into spinor frame variations by substituting for $\lambda^{\alpha}{}_{\beta}$ in terms of $u^{A}{}_{B}$.

1.	spinor frame	$\delta u_{A} = - u^{B}{}_{A} \, u_{B}$
2.	dual spinor frame	$\delta v^{A} = u^{A}{}_{B} \, v^{B}$
3.	conjugate spinor frame	$\delta \bar{u}_{\bar{A}} = - \bar{u}^{\bar{B}}{}_{\bar{A}} \, \bar{u}_{\bar{B}}$
4.	dual conjugate spinor frame	$\delta \bar{v}^{\bar{A}} = \bar{u}^{\bar{A}}{}_{\bar{B}} \, \bar{v}^{\bar{B}}$

(continued)

TABLE II.11 continued

5. spinor frame components of

a. spinor tensor $\qquad \delta\psi^{(A)} = R^{qs}_{pr}(u)^{(A)}{}_{(B)}\ \psi^{(B)}$

where

$$\psi^{(A)} = \psi^{A_1\cdots A_p}{}_{A_{p+1}\cdots A_{p+q}}{}^{\bar{A}_1\cdots\bar{A}_r}{}_{\bar{A}_{r+1}\cdots\bar{A}_{r+s}}$$

$$R^{qs}_{pr}(u)^{(A)}{}_{(B)} = \sum_{k=1}^{p}\left(\delta^{A_1}_{B_1}\cdots u^{A_k}{}_{B_k}\cdots\delta^{\bar{B}_{r+s}}_{\bar{A}_{r+s}}\right) - \sum_{k=1}^{q}\left(\delta^{A_1}_{B_1}\cdots u^{B_{p+k}}{}_{A_{p+k}}\cdots\delta^{\bar{B}_{r+s}}_{\bar{A}_{r+s}}\right)$$
$$+ \sum_{k=1}^{r}\left(\delta^{A_1}_{B_1}\cdots \bar{u}^{\bar{A}_k}{}_{\bar{B}_k}\cdots\delta^{\bar{B}_{r+s}}_{\bar{A}_{r+s}}\right) - \sum_{k=1}^{s}\left(\delta^{A_1}_{B_1}\cdots \bar{u}^{\bar{B}_{r+k}}{}_{\bar{A}_{r+k}}\cdots\delta^{\bar{B}_{r+s}}_{\bar{A}_{r+s}}\right)$$

b. spinor metric $\qquad \delta\varepsilon_{AB} = -u^{C}{}_{A}\,\varepsilon_{CB} - u^{C}{}_{B}\,\varepsilon_{AC} = 2\,u_{[AB]}$

SPECIAL CASES:

SL(2,C): $\qquad \varepsilon_{AB} = \varepsilon_{\circ AB} \qquad \delta\varepsilon_{AB} = 2\,u_{[AB]} = 0$

GL(2,C): $\qquad \varepsilon_{AB} = \phi\,\varepsilon_{\circ AB} \qquad \delta\varepsilon_{AB} = (\delta\phi)\,\varepsilon_{\circ AB} = \phi^{-1}\,(\delta\phi)\,\varepsilon_{AB} = 2\,u_{[AB]} = -u^{C}{}_{C}\,\varepsilon_{AB}$

c. spinor connection coefficients $\qquad \delta\Gamma^{A}{}_{Ba} = u^{A}{}_{C}\,\Gamma^{C}{}_{Ba} - u^{C}{}_{B}\,\Gamma^{A}{}_{Ca} - \partial_a u^{A}{}_{B} = -\nabla_a u^{A}{}_{B}$

where

$$\nabla_{\partial_a} u_B = \Gamma^{A}{}_{Ba}\,u_A$$

4. The Gravitational Variables: g, θ, Γ

In this section, I argue for using the metric, g, the soldering form, θ, and the connection, Γ, as the gravitational variables in a metric-connection theory. The argument is by analogy with the variables chosen in a gauge theory.

Recall that the following G-bundles are all required to be associated:

P = the principal G-bundle

E = the source field bundle,

TM = the tangent bundle,

$R_T(P)$ = the admissible tangent frame bundle,

T^*M = the cotangent bundle,

$R^*_T(P)$ = the admissible cotangent frame bundle

T^q_pM = the tangent tensor bundles, and

$R^q_p(P)$ = the admissible frame bundle for T^q_pM.

Further, if G is a spinor group, the spinor bundles ($T^{oo}_{\frac{1}{2}o}M$, $T^{\frac{1}{2}o}_{oo}M$, $T^{oo}_{o\frac{1}{2}}M$, $T^{o\frac{1}{2}}_{oo}M$, and $T^{qs}_{pr}M$) and the corresponding admissible spinor frame bundles ($R_S(P)$, $R^*_S(P)$, $\bar{R}_S(P)$, $\bar{R}^*_S(P)$, and $R^{qs}_{pr}(P)$) are also associated. To say that these bundles are associated, means that each bundle may be regarded as a G-bundle in such a way that they all have the same set of spacetime gauge patches, $U_\alpha \subset M$, and the same set of overlap gauge transformations,

$$\Lambda_{\alpha\beta} : U_\alpha \cap U_\beta \to G. \qquad (1)$$

A choice of gauge for any of these bundles is specified by a local cross section of the principal G-bundle, P:

$$\alpha : U_\alpha \to P. \qquad (2)$$

This induces local cross sections of all of the frame bundles:

$$\overset{\alpha}{e} : U_\alpha \to R_T(P), \qquad \overset{\alpha}{\theta} : U_\alpha \to R_T^*(P),$$
$$\overset{\alpha}{u} : U_\alpha \to R_S(P), \qquad \overset{\alpha}{v} : U_\alpha \to R_S^*(P), \tag{3}$$

etc. Each of these specifies the gauge on the corresponding vector bundle and all of the tensor bundles formed from that bundle. Thus, $\overset{\alpha}{e}$ or $\overset{\alpha}{\theta}$ specifies the gauge on all of the $T^q_p M$ including TM and T^*M, while $\overset{\alpha}{u}$ or $\overset{\alpha}{v}$ specifies the gauge on all of the $T^{qs}_{pr} M$ including all of the $T^q_p M$.

For each choice of spacetime gauge (say the α-gauge), the spacetime connection, Γ, determines a $\mathcal{L}G$-valued 1-form,

$$\overset{\alpha}{\Gamma} : TU_\alpha \to \mathcal{L}G, \tag{4}$$

where $\mathcal{L}G$ is the Lie algebra of G. On the overlap of two patches, $U_\alpha \cap U_\beta$, the connection 1-forms are related by

$$\overset{\alpha}{\Gamma} = \Lambda_{\alpha\beta} \overset{\beta}{\Gamma} \Lambda^{-1}_{\alpha\beta} + \Lambda_{\alpha\beta}\, d(\Lambda^{-1}_{\alpha\beta}). \tag{5}$$

The spacetime connection, Γ, is completely analogous to the gauge connection, A, and so it is reasonable to regard Γ as one of the gravitational potentials.

The property of the metric-connection theories which distinguishes them from the gauge theories is the fact that the frame bundle, $R_T(P)$, is contained in the general linear frame bundle, GL(M). Just how $R_T(P)$ sits within GL(M) is specified by the soldering form θ. For each choice of spacetime gauge (say the α-gauge), the soldering form, θ, becomes the corresponding 1-form frame field,

$$\overset{\alpha}{\vartheta} : U_\alpha \to R_T^*(P) \ . \tag{6}$$

On the overlap of two patches, $U_\alpha \cap U_\beta$, the soldering 1-forms are

related by

$$\overset{\alpha}{\theta}{}^{\mu} = (R_T \Lambda_{\alpha\beta})^{\mu}{}_{\nu} \, \overset{\beta}{\theta}{}^{\nu} \quad . \tag{7}$$

As a collection of four 1-forms, θ^{α} may also be regarded as an R^4-valued 1-form,

$$\overset{\alpha}{\theta} : TU_{\alpha} \to R^4 \quad . \tag{8}$$

Further, $R^4 = \mathcal{L}T(4,R)$ is the Lie algebra of the 4-dimensional real translation group, $T(4,R)$. Thus, for each choice of spacetime gauge, the soldering form, θ, determines a $\mathcal{L}T(4,R)$ - valued 1-form,

$$\overset{\alpha}{\theta} : TU_{\alpha} \to \mathcal{L}T(4,R). \tag{9}$$

This is analogous to equation (4) for the spacetime connection Γ, and to the corresponding equation for the gauge connection, A. However, θ differs from Γ and A because its transformation rule (7) is homogeneous whereas the transformation rule (5) for Γ has an inhomogeneous second term. Further, the transformation rule (7) involves the group, G, rather than the group, $T(4,R)$.

To improve the analogy between θ and Γ or A, let me first give a heuristic argument for regarding θ as the gauge potential for the translation group (or the coordinate transformation group or the diffeomorphism group). In constructing a covariant derivative one usually includes a compensating field for each generator of the gauge group. To include translations as well as rotations and gauge transformations, one might define the covariant derivative to be

$$\nabla_{\alpha} = \partial_{\alpha} + \lambda^{\gamma}{}_{\alpha} \, P_{\gamma} + \Gamma^{\gamma\delta}{}_{\alpha} \, J_{\gamma\delta} + A^{P}{}_{\alpha} \, T_{P} \, , \tag{10}$$

where P_γ, $J_{\gamma\delta}$ and T_P are the generators of translations, rotations and gauge transformations; and $\lambda^\gamma{}_\alpha$, $\Gamma^{\gamma\delta}{}_\alpha$ and $A^P{}_\alpha$ are the corresponding compensating fields. Recall however, that the generator of translations is $P_\gamma = \partial_\gamma$. Consequently, the first two terms of (10) combine into

$$e_\alpha = (\delta^\gamma{}_\alpha + \lambda^\gamma{}_\alpha)\partial_\gamma \ , \tag{11}$$

which may be taken as an orthonormal frame. Thus, although one started out trying to define a covariant derivative in a coordinate direction, one ends up with a covariant derivative in an orthonormal direction. Although, $\lambda^\gamma{}_\alpha$ is actually the compensating field, it transforms highly non-covariantly under both coordinate and frame transformations. Consequently, it is easier to use the tetrad components,

$$e_\alpha{}^a = \delta^a{}_\alpha + \lambda^a{}_\alpha \ , \tag{12}$$

or their inverses, $\theta^\alpha{}_a$, as the potential for the translation group.

The analogy between θ, Γ and A will be improved once I have discussed the properties of the connection, Γ, for various groups, G.

Recall (from Appendix B) that in a gauge, α, the covariant derivative of a field, ψ, in a direction, X, is

$$\nabla_X \overset{\alpha A}{\psi} = X(\overset{\alpha A}{\psi}) + [R\overset{\alpha}{\Gamma}(X)]^A{}_B \ \overset{\alpha B}{\psi} \ . \tag{13}$$

Here, $\overset{\alpha A}{\psi}$ are the components of ψ in the α-gauge and $\nabla_X \overset{\alpha A}{\psi}$ are the components of $\nabla_X \psi$ in the α-gauge. These transform under a representation, R, of the group, G. Further, $X(\overset{\alpha A}{\psi})$ is the directional derivative of $\overset{\alpha A}{\psi}$ in the direction, X, and finally, $[R\overset{\alpha}{\Gamma}(X)]^A{}_B$ are the matrix components of the representation, R, of the Lie algebra element, $\overset{\alpha}{\Gamma}(X)$, obtained when the connection 1-form in the α-gauge, $\overset{\alpha}{\Gamma}$, is evaluated on the tangent vector, X.

For a tangent vector field, Y, its components in the α-gauge, $\overset{\alpha}{Y}{}^{\mu}$, are its components relative to the admissible frame field, $\overset{\alpha}{e}_{\mu}$. These transform under the representation R_T. Thus the covariant derivative of Y in the direction X is

$$\nabla_X \overset{\alpha}{Y}{}^{\mu} = X(\overset{\alpha}{Y}{}^{\mu}) + [R_T \overset{\alpha}{\Gamma}(X)]^{\mu}{}_{\nu} \overset{\alpha}{Y}{}^{\nu} . \tag{14}$$

I emphasize that the differentiated vector, Y, *must* be expanded in an *admissible* basis, $\overset{\alpha}{e}_{\mu}$, whereas the differentiating vector, X, *may* be expanded in an *arbitrary* basis. When X and Y are both chosen as admissible basis vectors, one obtains the definition of the frame components of the connection:

$$\overset{\alpha}{\Gamma}{}^{\mu}{}_{\nu\lambda} = (\nabla_{\overset{\alpha}{e}_{\lambda}} \overset{\alpha}{e}_{\nu})^{\mu} = [R_T \overset{\alpha}{\Gamma}(\overset{\alpha}{e}_{\lambda})]^{\mu}{}_{\nu} . \tag{15}$$

On the other hand, if Y is chosen as the admissible basis vector, $\overset{\alpha}{e}_{\nu}$, whereas X is chosen as a coordinate basis vector, ∂_a, one obtains the definition of the mixed components of the connection:

$$\overset{\alpha}{\Gamma}{}^{\mu}{}_{\nu a} = (\nabla_{\partial_a} \overset{\alpha}{e}_{\nu})^{\mu} = [R_T \overset{\alpha}{\Gamma}(\partial_a)]^{\mu}{}_{\nu} . \tag{16}$$

These are related by

$$\overset{\alpha}{\Gamma}{}^{\mu}{}_{\nu\lambda} = \overset{\alpha}{e}{}_{\lambda}{}^{a} \overset{\alpha}{\Gamma}{}^{\mu}{}_{\nu a} . \tag{17}$$

For any spacetime symmetry group, G, its tangent representation, $R_T(G)$, is a subgroup of GL(4,R), and the tangent representation of its Lie algebra, $R_T(\mathcal{L}G)$, is a sub-Lie algebra of $\mathcal{L}GL(4,R) = M(4,R)$ = the set of 4 × 4 real matrices. (The relevant sub-Lie algebras of $\mathcal{L}GL(4,R)$ are listed in Table II.5.)

In particular, if $R_T(G) \subset O(3,1,R)$, then the admissible frames are orthonormal, so that the frame components of the metric are the constant Minkowski metric, $\overset{\alpha}{g}_{\mu\nu} = \eta_{\mu\nu}$. Further, $R_T(\mathcal{L}G) \subset \mathcal{L}O(3,1,R)$ consists of antisymmetric matrices when the index is lowered using $\eta_{\mu\nu}$; i.e. if $M^{\mu}{}_{\nu} \in \mathcal{L}O(3,1,R)$, then

$$M_{\mu\nu} + M_{\nu\mu} = \eta_{\mu\lambda} M^{\lambda}{}_{\nu} + \eta_{\nu\lambda} M^{\lambda}{}_{\mu} = 0. \tag{18}$$

Consequently, the covariant derivative of the metric vanishes,

$$\begin{aligned} \nabla_a \overset{\alpha}{g}_{\mu\nu} &= \partial_a \overset{\alpha}{g}_{\mu\nu} - \overset{\alpha\lambda}{\Gamma}{}_{\mu a} \overset{\alpha}{g}_{\lambda\nu} - \overset{\alpha\lambda}{\Gamma}{}_{\nu a} \overset{\alpha}{g}_{\mu\lambda} \\ &= - \overset{\alpha}{\Gamma}_{\nu\mu a} - \overset{\alpha}{\Gamma}_{\mu\nu a} = 0, \end{aligned} \tag{19}$$

so that Γ is a Cartan connection.

Conversely, if the covariant derivative of the metric vanishes, then by restricting to orthonormal frames, one reduces the tangent bundle to an $O(3,1,R)$ - bundle and reduces the connection to an $O(3,1,R)$ - connection.

Similarly, if $R_T(G) \subset CO(3,1,R)$, then the admissible frames are conformal orthonormal, the frame components of the metric are conformally Minkowski, $\overset{\alpha}{g}_{\mu\nu} = \overset{\alpha}{\Omega}\, \eta_{\mu\nu}$, and if $M^{\mu}{}_{\nu} \in R_T(\mathcal{L}G) \subset \mathcal{L}CO(3,1,R)$, then $M^{\mu}{}_{\nu}$ has no trace free symmetric part when the index is lowered using any conformally Minkowski metric:

$$M_{\mu\nu} + M_{\nu\mu} = \Omega\eta_{\mu\lambda} M^{\lambda}{}_{\nu} + \Omega\eta_{\nu\lambda} M^{\lambda}{}_{\mu} = \frac{1}{2} M^{\lambda}{}_{\lambda}\, \Omega\eta_{\mu\nu}\,. \tag{20}$$

(Here, Ω may or may not equal $\overset{\alpha}{\Omega}$.) Consequently, the covariant derivative of the metric is proportional to the metric,

$$\nabla_a \overset{\alpha}{g}_{\mu\nu} = \partial_a (\overset{\alpha}{\Omega}\, \eta_{\mu\nu}) - \overset{\alpha\lambda}{\Gamma}_{\mu a}\, \overset{\alpha}{\Omega}\, \eta_{\lambda\nu} - \overset{\alpha\lambda}{\Gamma}_{\nu a}\, \overset{\alpha}{\Omega}\, \eta_{\mu\lambda}$$

$$= (\overset{\alpha}{\Omega}{}^{-1}\, \partial_a \overset{\alpha}{\Omega} - \frac{1}{2}\, \overset{\alpha\lambda}{\Gamma}_{\lambda a}) \overset{\alpha}{g}_{\mu\nu}$$

$$= -\frac{1}{2}\, \overset{\alpha}{\lambda}_a\, \overset{\alpha}{g}_{\mu\nu}\,, \qquad (21)$$

so that Γ is a Weyl-Cartan connection. Conversely, for a Weyl-Cartan connection, restricting to conformal orthonormal frames reduces the tangent bundle to a $CO(3,1,R)$ - bundle and reduces the connection to a $CO(3,1,R)$ - connection.

Similarly again, if $R_T(G) \subset VL(4,R)$, then the admissible frames are unit volume, the determinant of the frame components of the metric is $\hat{g} = -1$, and any $M^{\mu}{}_{\nu} \in R_T(\mathcal{L}G) \subset \mathcal{L}VL(4,R)$ is trace-free, $M^{\lambda}{}_{\lambda} = 0$. Consequently, the covariant derivative of the metric is trace-free,

$$\overset{\alpha\mu\nu}{g}\, \nabla_a \overset{\alpha}{g}_{\mu\nu} = \hat{g}^{-1}\, \partial_a \hat{g} - 2\, \overset{\alpha\lambda}{\Gamma}_{\lambda a} = 0. \qquad (22)$$

(This equation might be called the anti-Weyl-compatibility condition.) Conversely, if the covariant derivative of the metric is trace-free, then restricting to unit volume frames reduces the tangent bundle to a $VL(4,R)$ - bundle and reduces the connection to a $VL(4,R)$ - connection.

Finally, if $R_T(G) = GL(4,R)$, then any frame is admissible, the frame components of the metric are arbitrary, the Lie algebra, $R_T(\mathcal{L}G)$, is all of $\mathcal{L}GL(4,R) = M(4,R)$, the covariant derivative of the metric is unrestricted, and the connection is completely general.

If the spacetime symmetry group, G, also has a spinor representation, $R_S(G) \subset GL(2,C)$, then the connection can also be classified by the value of the covariant derivative of the spinor metric, $\nabla_a \overset{\alpha}{\varepsilon}_{AB}$. Relative to an

admissible spinor frame $\overset{\alpha}{u}_A$, the <u>components of the spinor connection</u> are defined as

$$\overset{\alpha A}{\Gamma}_{Ba} = (\nabla_{\partial_a} \overset{\alpha}{u}_B)^A = [R_S \overset{\alpha}{\Gamma}(\partial_a)]^A{}_B \ . \tag{23}$$

Since $\nabla_a \overset{\alpha}{\varepsilon}_{AB}$ is antisymmetric in A and B, it must be proportional to $\overset{\alpha}{\varepsilon}_{AB}$:

$$\nabla_a \overset{\alpha}{\varepsilon}_{AB} = \overset{\alpha}{X}_a \overset{\alpha}{\varepsilon}_{AB} \ . \tag{24}$$

If $R_S(\mathcal{L}G) \subset \mathcal{L}SL(2,C)$, then $\overset{\alpha A}{\Gamma}_{Aa} = 0$ and $\overset{\alpha}{X}_a = 0$. If $R_S(\mathcal{L}G) \subset \mathcal{L}CL(2,C)$, then $\overset{\alpha A}{\Gamma}_{Aa}$ and $\overset{\alpha}{X}_a = -\frac{1}{4}\overset{\alpha}{\lambda}_a$ are pure real where $\overset{\alpha}{\lambda}_a$ is the Weyl-potential of equation (21). If $R_S(\mathcal{L}G) \subset \mathcal{L}PL(2,C)$, then $\overset{\alpha A}{\Gamma}_{Aa} = 2\,i\,\overset{\alpha}{A}_a$ and $\overset{\alpha}{X}_a$ are pure imaginary where $\overset{\alpha}{A}_a$ may be identified as an electromagnetic potential. Finally, if $R_S(\mathcal{L}G) = \mathcal{L}GL(2,C)$, then $\overset{\alpha A}{\Gamma}_{Aa}$ and X_a are complex.

The above classification of Γ by the values of ∇g or $\nabla \varepsilon$ completely characterizes Γ for each of the spacetime symmetry groups, G, listed in Tables II.2 and II.4. These are all homogeneous groups. What about the inhomogeneous versions of these groups?

Recall that an inhomogeneous tangent group is the semi-direct product

$$G = H \times_{R_T} T(4,R), \tag{25}$$

where H is the corresponding homogeneous tangent group, T(4,R) is the 4-dimensional real translation group, R_T is the defining representation of H acting on $T(4,R) = R^4$, and the product is defined by

$$(\Lambda,a) \circ (M,b) = (\Lambda M,\ \Lambda b + a). \tag{26}$$

The Lie algebra of G is the semi-direct sum

$$\mathcal{L}G = \mathcal{L}H +_{R_T} \mathcal{L}T(4,R), \tag{27}$$

where the sum is componentwise,

$$(\lambda,\delta) + (\mu,\varepsilon) = (\lambda+\mu,\ \delta+\varepsilon), \tag{28}$$

and the Lie bracket is defined by

$$[\ (\lambda,\delta)\ ,\ (\mu,\varepsilon)\ ] = (\ [\lambda,\mu],\ \lambda\varepsilon - \mu\delta). \tag{29}$$

There is a 5-dimensional faithful representation of G and $\mathcal{L}G$ in which $(\Lambda,a) \in G$ is represented by the block matrix, $\begin{pmatrix} \Lambda & a \\ 0 & 1 \end{pmatrix}$, and $(\lambda,\varepsilon) \in \mathcal{L}G$ is represented by $\begin{pmatrix} \lambda & \varepsilon \\ 0 & 0 \end{pmatrix}$. Notice that for group elements the matrix product coinsides with (26), while for Lie algebra elements the matrix sum coincides with (28) and the matrix commutator coincides with (29). Further, the matrix exponential of a Lie algebra element is a group element. From now on I identify this representation with G and $\mathcal{L}G$. For future use notice that the inverse of a group element is

$$\begin{pmatrix} \Lambda & a \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \Lambda^{-1} & -\Lambda^{-1}a \\ 0 & 1 \end{pmatrix} . \tag{30}$$

Recall from Section 3 that the principal bundle for the inhomogeneous tangent group, G, is the affine tangent frame bundle

$$P = Q \times TM \quad , \tag{31}$$

where Q is the corresponding tangent frame bundle for the homogeneous tangent group, H. A choice of gauge is now a local cross section of P; i.e. a local affine tangent frame field, $(\overset{\alpha}{e}\ ,\ \overset{\alpha}{o})$, where

$$\overset{\alpha}{e} : U_\alpha \to Q, \tag{32}$$

is an admissible tangent frame field and

$$\overset{\alpha}{o} : U_\alpha \to TU_\alpha, \tag{33}$$

is a vector field regarded as a new origin for TM. Two affine frame fields are related by the overlap gauge transformations,

$$\underline{\Lambda}_{\alpha\beta} = \begin{pmatrix} \Lambda_{\alpha\beta} & a_{\alpha\beta} \\ 0 & 1 \end{pmatrix} : U_\alpha \cap U_\beta \to G \quad , \tag{34}$$

according to

$$\begin{aligned} (\overset{\alpha}{e}\ ,\ \overset{\alpha}{o}) &= (\overset{\beta}{e}\ ,\ \overset{\beta}{o})\underline{\Lambda}^{-1}_{\alpha\beta} \\ &= (\overset{\beta}{e},\ \overset{\beta}{o}) \begin{pmatrix} \Lambda^{-1}_{\alpha\beta} & -\Lambda^{-1}_{\alpha\beta}\, a_{\alpha\beta} \\ 0 & 1 \end{pmatrix} \\ &= (\overset{\beta}{e}\,\Lambda^{-1}_{\alpha\beta}\ ,\ \overset{\beta}{o} - \overset{\beta}{e}\,\Lambda^{-1}_{\alpha\beta}\, a_{\alpha\beta}). \end{aligned} \tag{35}$$

In other words

$$\overset{\alpha}{e}_\mu = \overset{\beta}{e}_\nu (\Lambda^{-1}_{\alpha\beta})^\nu{}_\mu, \tag{36}$$

$$\overset{\alpha}{o} = \overset{\beta}{o} - \overset{\beta}{e}_\nu (\Lambda^{-1}_{\alpha\beta})^\nu{}_\mu\ (a_{\alpha\beta})^\mu. \tag{37}$$

In the α-gauge any tangent vector, $X \in TU_\alpha$, may be specified by its <u>affine components</u>, $\overset{\alpha}{X}{}^\mu$, determined from

$$X = \overset{\alpha}{X}{}^\mu\ \overset{\alpha}{e}_\mu + \overset{\alpha}{o}. \tag{38}$$

Using (36) and (37), one finds that under the gauge transformation (34), the affine components of X undergo the affine transformation,

$$\overset{\alpha}{X}{}^\mu = (\Lambda_{\alpha\beta})^\mu{}_\nu\ \overset{\beta}{X}{}^\nu + (a_{\alpha\beta})^\mu. \tag{39}$$

Thus TM may be regarded as a G-bundle associated to P. (Note: With affine components, TM is not a G-vector bundle and so it is not possible to define a G-covariant derivative. However, it is still possible to define a G-connection.)

In the α-gauge, the G-connection, $\underline{\Gamma}$, determines an $\mathcal{L}G$-valued 1-form,

$$\overset{\alpha}{\underline{\Gamma}} = \begin{pmatrix} \overset{\alpha}{\Gamma} & \overset{\alpha}{\gamma} \\ 0 & 0 \end{pmatrix} \quad : \quad TU_\alpha \to \mathcal{L}G, \tag{40}$$

where

$$\overset{\alpha}{\Gamma} : TU_\alpha \to \mathcal{L}H, \tag{41}$$

$$\overset{\alpha}{\gamma} : TU_\alpha \to \mathcal{L}T(4,R) = R^4. \tag{42}$$

Under the gauge transformation (34), the connection 1-forms transform according to

$$\overset{\alpha}{\underline{\Gamma}} = \underline{\Lambda}_{\alpha\beta}\, \overset{\beta}{\underline{\Gamma}}\, \underline{\Lambda}_{\alpha\beta}^{-1} + \underline{\Lambda}_{\alpha\beta}\, d(\underline{\Lambda}_{\alpha\beta}^{-1}). \tag{43}$$

Using (40), (34), and (30) one finds

$$\overset{\alpha}{\Gamma} = \Lambda_{\alpha\beta}\, \overset{\beta}{\Gamma}\, \Lambda_{\alpha\beta}^{-1} + \Lambda_{\alpha\beta} d(\Lambda_{\alpha\beta}^{-1}), \tag{44}$$

$$\overset{\alpha}{\gamma} = \Lambda_{\alpha\beta}\, [\overset{\beta}{\gamma} - d(\Lambda_{\alpha\beta}^{-1}\, a_{\alpha\beta}) - \overset{\beta}{\Gamma}\, \Lambda_{\alpha\beta}^{-1}\, a_{\alpha\beta}]\ . \tag{45}$$

If one restricts attention to affine frame fields, $(\overset{\alpha}{e}, \overset{\alpha}{o})$, for which the origin vector field, $\overset{\alpha}{o}$ in (33), is the zero cross section of TM, then the restricted class of gauge transformations (34) have $a_{\alpha\beta} = 0$. This reduces TM from a G-bundle to an H-vector bundle. However, as long as $\overset{\alpha}{\gamma}$ is non-zero, the G-connection, $\overset{\alpha}{\underline{\Gamma}}$, is not reducible. Rather, it decomposes into the H-connection, $\overset{\alpha}{\Gamma}$ in (41), which transforms according to (44), and the R^4-valued 1-form, $\overset{\alpha}{\gamma}$ in (42), which transforms according to

$$\overset{a}{\gamma}{}^{\mu} = (\Lambda_{\alpha\beta})^{\mu}{}_{\nu}\,\overset{\beta}{\gamma}{}^{\nu}, \tag{46}$$

when one sets $a_{\alpha\beta} = 0$ in (45). Notice that the collection of 1-forms, γ, behaves exactly like the soldering form, θ. (Compare equation (42) with (9) and equation (46) with (7).) Thus it is reasonable and possible to identify γ with θ, although it is not necessary.

A G-connection, $\underline{\Gamma}$, is called an affine tangent connection if $\overset{\alpha}{\gamma}{}^{\mu}{}_{a} = \overset{\alpha}{\theta}{}^{\mu}{}_{a}$ whenever the origin vector field, $\overset{\alpha}{o}$, vanishes. On the other hand, a G-connection, $\underline{\Gamma}$, is called a generalized affine tangent connection if $\overset{\alpha}{\gamma}$ is unrelated to $\overset{\alpha}{\theta}$. In this context the H-connection, Γ, is called a linear tangent connection. (Thus the name "affine connection" when applied to the usual connection on the tangent bundle is a misnomer.) See Kobayashi and Nomizu [1963] ch. III for a more detailed discussion of linear connections, affine connections, and generalized affine connections.

It is obvious that a similar construction can be done which unifies the soldering form, θ, with a linear spinor connection, Γ, for a homogeneous spinor group, H, into an affine spinor connection, $\underline{\Gamma}$, for an affine spinor, $G = H \times_{R_T} T(4,R)$. One might also investigate a generalized affine tangent connection or a generalized affine spinor connection, but I do not understand the physical significance of the additional affine components, γ, in the connection. Nor do I understand the significance of the extra components of a connection for an inhomogeneous spinor group, $G = H \times_{R_S} T(2,C)$.

This completes my justification for using the soldering form, θ, as a gravitational variable in a metric-connection theory. To summarize, the soldering form, θ, and the linear connection, Γ, may be unified into an affine connection, $\underline{\Gamma}$, which is analogous to the gauge connection, A, in a gauge theory. However, in practice, the affine notation is cumbersome and unfamiliar. So I use θ and Γ as separate variables rather than $\underline{\Gamma}$.

What about the metric, $g_{\mu\nu}$? The spacetime metric is analogous to the inner product, ϕ_{jk}, introduced in the discussion of gauge theories in Section 1. In the gauge theory case, $\overset{\alpha}{\phi}_{jk}$ was used to define certain choices of gauge (by choosing an orthonormal frame field) or even to define the gauge group (by restricting to only orthonormal frames). Similarly, in Section 3, the frame components of the metric, $\overset{\alpha}{g}_{\mu\nu}$, were often used to define the admissible frame fields and the principal G-bundle, P.

In the gauge theory case, ϕ_{jk} was used to construct a scalar Lagrangian. Likewise $g_{\mu\nu}$ is used to construct a scalar Lagrangian. There are two differences. First, differentiations are performed in spacetime directions, not "gauge directions." All derivative indices are converted to admissible frame indices by contracting with the frame components, $e_{\alpha}{}^{a}$. Then all other contractions are performed using the frame components of the metric, $g_{\mu\nu}$. Second, in varying the Lagrangian, L, one actually varies the Lagrangian density

$$\mathcal{L} = \sqrt{-\tilde{g}}\, L = \sqrt{-\hat{g}}\, \theta\, L. \tag{47}$$

Thus, in varying $g_{\mu\nu}$ one must also vary $\sqrt{-\hat{g}}$, and in varying $\theta^{\alpha}{}_{a}$ one must also vary θ.

In the gauge theory case, one most often restricts attention to groups and representations which have invariant inner products. In that case, ϕ_{jk} is constant in position and independent of gauge, and so cannot be varied. Similarly, for the Lorentz group, O(3,1,R), and its subgroups, the admissible frames are orthonormal and the frame components of the metric are the components of the Minkowski metric, $g_{\mu\nu} = \eta_{\mu\nu}$. Since the Minkowski metric is constant and Lorentz invariant, it also cannot be varied in the Lagrangian.

However, for the group, GL(4,R), there is no invariant metric on the tangent bundle. Instead, one introduces the frame components of the metric, $g_{\mu\nu}$, as additional dynamic variables. This induces metrics on all of the tangent tensor bundles, including the bundle, T^1_1M, whose fibre is $\mathcal{L}GL(4,R)$. (Note: one does not have to use the standard induced metric.) In Section 5, I discuss the kinetic Lagrangian for $g_{\mu\nu}$ (as well as $\theta^\mu{}_a$ and $\Gamma^\mu{}_{\nu a}$)and point out that I regard the Lagrangian (II.5.19) as most analogous to a Yang-Mills Lagrangian.

Like ϕ_{jk}, the spacetime metric, $g_{\mu\nu}$, behaves much like a Goldstone-Higgs field. By restricting to orthonormal frames, one reduces the tangent tensor bundles from GL(4,R) - bundles to O(3,1,R) - bundles. However, as long as $\nabla_a g_{\mu\nu} \neq 0$, the GL(4,R) - connection does not reduce. Rather, it decomposes into an O(3,1,R) - connection and 10 residual 1-form fields. The symmetry is broken. For Lagrangian (II.5.19), except for special values of the coupling constants, the 10 residual 1-form fields become massive and absorb the 10 components of $g_{\mu\nu}$ as their longitudinal components.

What about the spacetime symmetry groups other than GL(4,R) and O(3,1,R)? For CO(3,1,R), the admissible frames are conformal orthonormal, $g_{\mu\nu} = \Omega\, \eta_{\mu\nu}$, and $\nabla_a g_{\mu\nu} = -\frac{1}{2}\, \lambda_a\, g_{\mu\nu}$, where $\lambda_a = \Gamma^\mu{}_{\mu a} - 2\, \Omega^{-1}\, \partial_a \Omega$. In varying the metric, one can only vary the conformal factor, Ω. Upon restricting to orthonormal frames, so that $\Omega = 1$, the symmetry is broken down to O(3,1,R) and the residual 1-form field is $\lambda_a = \Gamma^\mu{}_{\mu a}$. If one uses Lagrangian (II.5.19), then the $\nabla_a g_{\mu\nu}$ terms become a mass term for λ_a.

Similarly, for VL(4,R) or SL(4,R) the metric has unit determinant, $\hat{g} = -1$, and its covariant derivative is trace-free, $g^{\mu\nu}\, \nabla_a g_{\mu\nu} = 0$. One must vary the metric subject to the constraint, $\hat{g} = -1$. Upon restricting to orthonormal frames, the symmetry is again reduced to O(3,1,R) and there are

9 residual 1-form fields which are the independent components of $\nabla_a g_{\mu\nu}$.

For the spinor groups, one should technically use the spinor metric, ε_{AB}, as the dynamic field rather than $g_{\mu\nu}$. This is analogous to the gauge theory recommendation that the dynamic metrics should be the metrics on the fundamental representations. However, for SL(2,C), the spinor metric has the constant value, $\varepsilon_{AB} = \varepsilon_{\circ AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and so is non-dynamic. For VL(2,C), the spinor metric, $\varepsilon_{AB} = \pm\, \varepsilon_{\circ AB}$, is again non-dynamic. For CL(2,C), the spinor metric is $\varepsilon_{AB} = \phi\, \varepsilon_{\circ AB}$ with ϕ real and positive, while the tangent metric is $g_{\mu\nu} = \phi^2\, \eta_{\mu\nu}$. Consequently, $g_{\mu\nu}$ completely determines ε_{AB} and can be used as the dynamic variable. For RL(2,C), $\varepsilon_{AB} = \phi\, \varepsilon_{\circ AB}$ with ϕ real, while $g_{\mu\nu} = \phi^2\, \eta_{\mu\nu}$. Hence, $g_{\mu\nu}$ only determines ε_{AB} up to sign but can probably still be used as the dynamic variable. On the other hand, for PL(2,C) and GL(2,C), the tangent metric, $g_{\mu\nu} = |\phi|^2\, \eta_{\mu\nu}$, only determines the spinor metric, $\varepsilon_{AB} = \phi\, \varepsilon_{\circ AB}$, up to a phase which can vary continuously. Consequently, one must use ε_{AB} as the dynamic variable. More work is needed on this special case.

I have now completed my demonstration that in the same sense as ϕ_{jk} may be regarded as a gauge potential in addition to $A^P{}_a$, so the metric, $g_{\mu\nu}$, or the spinor metric, ε_{AB}, may be regarded as a gravitational potential along with $\Gamma^\mu{}_{\nu a}$ and $\theta^\alpha{}_a$. In the remainder of this section, I discuss the gravitational fields or "curvatures" constructed from the gravitational potentials or "connections," g, θ and Γ.

From the spacetime connection, $\Gamma^\alpha{}_{\beta a}$, one constructs the spacetime curvature,

$$\hat{R}^\alpha{}_{\beta ab} = \partial_a \Gamma^\alpha{}_{\beta b} - \partial_b \Gamma^\alpha{}_{\beta a} + \Gamma^\alpha{}_{\gamma a}\, \Gamma^\gamma{}_{\beta b} - \Gamma^\alpha{}_{\gamma b}\, \Gamma^\gamma{}_{\beta a} \quad . \tag{48}$$

This is obviously analogous to the gauge curvature, F^{P}_{ab}, constructed from the gauge connection, A^{P}_{a}.

What is the "curvature" for θ^{α}_{a}? From the heuristic point of view of equation (10), in which $\lambda^{a}_{\alpha} = e_{\alpha}^{\ a} - \delta^{a}_{\alpha}$ is regarded as a compensating field for the translation group, the "curvature" may be taken as the commutator functions,

$$c^{a}_{\alpha\beta} = e_{\gamma}^{\ a}\, c^{\gamma}_{\alpha\beta} = e_{\alpha}^{\ b}\partial_{b}e_{\beta}^{\ a} - e_{\beta}^{\ b}\partial_{b}e_{\alpha}^{\ a}$$

$$= e_{\alpha}^{\ b}\partial_{b}\lambda^{a}_{\ \beta} - e_{\beta}^{\ b}\partial_{b}\lambda^{a}_{\ \alpha} \quad . \tag{49}$$

There are two objections to using $c^{a}_{\alpha\beta}$ as the "curvature" for the translation group. First, $c^{a}_{\alpha\beta}$ is not a tensor. This objection can be rationalized away by pointing out that unlike A^{P}_{a} and $\Gamma^{\alpha}_{\beta a}$, the frame, θ^{α}_{a} is a tensor; so its curvature need not be. Second, $c^{a}_{\alpha\beta}$ does not take into account the interaction between the translation and rotation groups.

Both of these objections are eliminated by using the torsion,

$$Q^{\alpha}_{ab} = \theta^{\gamma}_{\ a}\,\theta^{\beta}_{\ b}\,(-c^{\alpha}_{\gamma\beta} + \Gamma^{\alpha}_{\beta\gamma} - \Gamma^{\alpha}_{\gamma\beta})$$

$$= \partial_{a}\theta^{\alpha}_{\ b} - \partial_{b}\theta^{\alpha}_{\ a} + \Gamma^{\alpha}_{\gamma a}\,\theta^{\gamma}_{\ b} - \Gamma^{\alpha}_{\gamma b}\,\theta^{\gamma}_{\ a} \quad , \tag{50}$$

as the "curvature" corresponding to the frame, θ^{α}_{a}. The best justification for using Q^{α}_{ab} comes from noting that an affine connection,

$$\underline{\Gamma}_{a} = \begin{pmatrix} \Gamma^{\alpha}_{\ \beta a} & \theta^{\alpha}_{\ a} \\ 0 & 0 \end{pmatrix} \quad , \tag{51}$$

can be used to construct an affine curvature,

$$\hat{\underline{R}}_{ab} = \partial_a \underline{\Gamma}_b - \partial_b \underline{\Gamma}_a + \underline{\Gamma}_a \underline{\Gamma}_b - \underline{\Gamma}_b \underline{\Gamma}_a$$

$$= \partial_a \begin{pmatrix} \Gamma^{\alpha}{}_{\beta b} & \theta^{\alpha}{}_{b} \\ 0 & 0 \end{pmatrix} - \partial_b \begin{pmatrix} \Gamma^{\alpha}{}_{\beta a} & \theta^{\alpha}{}_{a} \\ 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} \Gamma^{\alpha}{}_{\gamma a} & \theta^{\alpha}{}_{a} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \Gamma^{\gamma}{}_{\beta b} & \theta^{\gamma}{}_{b} \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \Gamma^{\alpha}{}_{\gamma b} & \theta^{\alpha}{}_{b} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \Gamma^{\gamma}{}_{\beta a} & \theta^{\gamma}{}_{a} \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \hat{R}^{\alpha}{}_{\beta ab} & Q^{\alpha}{}_{ab} \\ 0 & 0 \end{pmatrix} , \qquad (52)$$

whose components are just the linear curvature, $\hat{R}^{\alpha}{}_{\beta ab}$, and the torsion, $Q^{\alpha}{}_{ab}$.

Finally, the "curvature" corresponding to the metric, $g_{\alpha\beta}$, may be taken either as its partial derivative, $\partial_a g_{\alpha\beta}$, or as its covarient derivative, $\nabla_a g_{\alpha\beta}$. I prefer the tensor, $\nabla_a g_{\alpha\beta}$.

5. The Action: $S[\psi,g,\theta,\Gamma,A]$

In this section, I study the action functional, $S[\psi,g,\theta,\Gamma,A]$, of a metric-connection theory with spacetime symmetry group, G_1, which is also a local gauge theory with internal gauge group, G_2. The discussion is completely analogous to the discussion of the action for an internal gauge theory at the end of Section 1.

The action, S, must be invariant under both spacetime gauge transformations and internal gauge transformations. It is usually assumed that the action is local; i.e. that it may be written as

$$S[\psi,g,\theta,\Gamma,A] = \int L \sqrt{-\tilde{g}}\, d^4x, \tag{1}$$

where the Lagrangian,

$$L = L(\psi,\ldots,\partial^{(m)}\psi, g,\ldots,\partial^{(i)}g, \theta,\ldots,\partial^{(j)}\theta, \Gamma,\ldots,\partial^{(k)}\Gamma, A,\ldots,\partial^{(n)}A), \tag{2}$$

is a strictly local function of ψ, g, θ, Γ, A, and a finite number of their derivatives. (To shorten notation, I will not write the derivatives in most future expressions for the Lagrangian.) It is also usually assumed that the Lagrangian, L, is a scalar under coordinate transformations, spacetime symmetry transformations, and internal gauge transformations. I make both of these assumptions throughout the thesis.

A metric-connection theory is called <u>special relativistic</u> or <u>global</u> if the connection, $\Gamma^{\alpha}{}_{\beta c}$, is required to be metric-compatible, torsion-free and flat:

$$\nabla_c g_{\alpha\beta} = 0, \qquad Q^{\alpha}{}_{bc} = 0, \qquad \hat{R}^{\alpha}{}_{\beta cd} = 0. \tag{3}$$

Regarding the covariant derivative of the metric, $\nabla_c g_{\alpha\beta}$, and the torsion, $Q^{\alpha}{}_{bc}$, as the "curvatures" for the metric, $g_{\alpha\beta}$, and frame, $\theta^{\alpha}{}_{b}$, one might say that a special relativistic theory must have vanishing "curvatures" for $g_{\alpha\beta}$, $\theta^{\alpha}{}_{b}$, and $\Gamma^{\alpha}{}_{\beta c}$.

A theory is special relativistic iff it is always possible to find a choice of admissible frame (called an inertial frame) and a choice of coordinates (called inertial coordinates) in which the connection vanishes, the frame is the coordinate frame, and the metric components are constant:

$$g_{\alpha\beta} = \text{const}, \qquad \theta^{\alpha}{}_{b} = \delta^{\alpha}{}_{b}, \qquad \Gamma^{\alpha}{}_{\beta c} = 0. \tag{4}$$

Such a choice of coordinates and admissible frame is determined uniquely up to a coordinate transformation of the form,

$$x'^{\mu'} = \Lambda^{\mu'}{}_{\nu}\ x^{\nu} + a^{\mu}\ , \tag{5}$$

and the corresponding coordinate frame transformation,

$$\partial_{\mu'} = (\Lambda^{-1})^{\nu}{}_{\mu'}\ \partial_{\nu}\ , \tag{6}$$

where the a^{μ} are four constants and the $\Lambda^{\mu'}{}_{\nu}$ form a constant matrix belonging to the representation, $R_T(G_1)$, of the spacetime symmetry group, G_1. For example, if $G_1 = CO(3,1,R)$ then it is possible to find a coordinate system for which the coordinate frame is conformal orthonormal and covariantly constant and further the conformal factor of the metric components is a constant. Such a coordinate system is determined up to a constant Lorentz transformation, a constant dilation and a constant translation.

Notice that equations (4) can also be written as

$$\partial_c g_{\alpha\beta} = 0, \qquad c^{\alpha}{}_{\beta\gamma} = 0, \qquad \Gamma^{\alpha}{}_{\beta c} = 0. \tag{7}$$

Consequently, if one would rather regard the non-covariant quantities, $\partial_c g_{\alpha\beta}$ and $c^{\alpha}{}_{\beta\gamma}$, as the "curvatures" for $g_{\alpha\beta}$ and $\theta^{\alpha}{}_{b}$, then one can also say that a theory is special relativistic iff there exists an admissible frame field in which the "curvatures," $\partial_c g_{\alpha\beta}$, $c^{\alpha}{}_{\beta\gamma}$, and $\hat{R}^{\alpha}{}_{\beta cd}$, vanish.

To obtain a special relativistic theory, the constraints (3) can be imposed in the Lagrangian by using Lagrange multipliers. Alternatively and equivalently, they can be imposed by using inertial coordinates and frames in the action (1), so that equations (4) are satisfied. In that case, one only varies ψ and A in the special relativistic action,

$$S[\psi,A] = S[\psi,g^{\circ}_{\alpha\beta},\delta^{\alpha}{}_{b},0,A]$$

$$= \int L(\psi,g^{\circ}_{\alpha\beta},\delta^{\alpha}{}_{b},0,A)\sqrt{-\tilde{g}^{\circ}}d^{4}x \quad , \tag{8}$$

where $g^{\circ}_{\alpha\beta}$ denotes the constant value of the metric.

One might also be interested in a theory in which only one or two of the conditions (3) are required to be satisfied. If one requires the connection to be metric-compatible and torsion-free but not flat, then the metric-connection theory reduces to a metric theory. If only the covariant derivative of the metric is required to vanish, one obtains a metric-Cartan connection theory; while if only the torsion vanishes, one obtains a metric-connection theory with a non-metric-compatible connection. On the other hand if the curvature vanishes but ∇g and/or Q does not, then one obtains a partially special relativistic theory with non-metricity and/or torsion.

Any metric-connection theory which is not special relativistic is called general relativistic or local. Thus a local metric-connection theory may have some classical solutions in which equations (3) are satisfied but must also have some solutions in which they are not satisfied.

The Lagrangian (2) for a local metric-connection theory may be decomposed as follows: First there is a matter Lagrangian, L_M, obtained by setting g, θ and Γ to their special relativistic values (4) and adjusting the constant term:

$$L_M(\psi,A) = L(\psi,\ g^{\circ}_{\alpha\beta},\ \delta^{\alpha}{}_{b},\ 0,\ A) + \frac{\hbar c}{8\pi L^2}\ \Lambda\ . \tag{9}$$

It is usually assumed that the energy density of the matter Lagrangian has a minimum. The constant, Λ, is adjusted so that the minimum energy is zero and the energy density is positive definite. The matter Lagrangian is regarded as the Lagrangian for the special relativistic limit of the theory. It may be decomposed as in Section 1:

$$L_M(\psi,A) = L_A(A) + L_S(\psi) + L_I(\psi,A) + L_C\ , \tag{10}$$

where

$$L_C = L_M(0,0)\ , \tag{11}$$

$$L_S(\psi) = L_M(\psi,0) - L_C\ , \tag{12}$$

$$L_A(A) = L_M(0,A) - L_C\ . \tag{13}$$

Equations (9) and (11) show that the constant term in the full Lagrangian is

$$L(0,g^{\circ}_{\alpha\beta},\ \delta^{\alpha}{}_{b},0,0) = -\frac{\hbar c}{8\pi L^2}\ \Lambda + L_C\ . \tag{14}$$

Next there is a gravitational Lagrangian, L_G, obtained by setting $\psi = 0$ and $A = 0$ and again adjusting the constant term:

$$L_G(g,\theta,\Gamma) = L(0,g,\theta,\Gamma,0) - L_C. \tag{15}$$

The gravitational Lagrangian is the Lagrangian for the corresponding matter-free or vacuum metric-connection theory. Equations (14) and (15) show that the constant term in the gravitational Lagrangian is

$$L_G(g^\circ_{\alpha\beta}, \delta^\alpha{}_b, 0) = -\frac{\hbar c}{8\pi L^2}\Lambda \ . \tag{16}$$

Thus Λ may be interpreted as a cosmological constant.

Finally, the remainder is the gravitational interaction Lagrangian, L_I', defined so that the full Lagrangian is

$$L(\psi, g, \theta, \Gamma, A) = L_G(g, \theta, \Gamma) + L_M(\psi, A) + L_I'(\psi, g, \theta, \Gamma, A). \tag{17}$$

The sum $L_M + L_I'$ is the interacting matter Lagrangian, while $L_G + L_I'$ is the interacting gravitational Lagrangian.

I postpone a detailed discussion of minimal coupling to Section III.4. It suffices here to say that a metric-connection theory is minimally coupled to the gravitational field if the interacting matter Lagrangian, $L_M + L_I'$, can be obtained from the (special relativistic) matter Lagrangian, L_M, by a specific procedure given in Section III.4.

The choice of gravitational Lagrangian, L_G, is the topic of Chapter V. I here discuss the choice of gravitational Lagrangian which I consider to be most analogous to the Yang-Mills Lagrangian (II.1.31).

If one regards the frame components of the metric, $g_{\alpha\beta}$, the coordinate components of the 1-form frame, $\theta^\alpha{}_a$, and the mixed components of the connection, $\Gamma^\alpha{}_{\beta a}$, as the gravitational potentials analogous to the gauge potentials, $A^P{}_a$, and regards the covariant derivative of the metric, $\nabla_a g_{\alpha\beta}$, the torsion, $Q^\alpha{}_{ab}$, and the full curvature, $\hat{R}^\alpha{}_{\beta ab}$, as the corresponding gravitational fields analogous to the gauge fields, $F^P{}_{ab}$, then the gravitational Lagrangian most analogous to the Yang-Mills Lagrangian is

$$L_G = - s \frac{\hbar c}{16\pi L_1^2} g^{ab} g^{\alpha\gamma} g^{\beta\delta} (\nabla_a g_{\alpha\beta})(\nabla_b g_{\gamma\delta})$$

$$- s \frac{\hbar c}{16\pi L_2^2} g_{\alpha\beta} g^{ac} g^{bd} Q^{\alpha}{}_{ab} Q^{\beta}{}_{cd}$$

$$- \frac{\hbar c}{16\pi\alpha_G} g^{ac} g^{bd} \hat{R}^{\alpha}{}_{\beta ab} \hat{R}^{\beta}{}_{\alpha cd} , \qquad (18)$$

where α_G is a dimensionless coupling constant and L_1 and L_2 are coupling constants with the dimensions of length. One could also introduce separate coupling constants for the trace part of $\nabla_a g_{\alpha\beta}$, and for the anti-symmetric, symmetric and trace parts of $\hat{R}^{\alpha}{}_{\beta ab}$:

$$L_G = - s \frac{\hbar c}{16\pi L_1^2} g^{\alpha\gamma} g^{\beta\delta} (\nabla_a g_{\alpha\beta})(\nabla^a g_{\gamma\delta}) - s \frac{\hbar c}{16\pi L_1'^2} g^{\alpha\beta} g^{\gamma\delta} (\nabla_a g_{\alpha\beta})(\nabla^a g_{\gamma\delta})$$

$$- s \frac{\hbar c}{16\pi L_2^2} Q^{\alpha}{}_{ab} Q_{\alpha}{}^{ab} - \frac{\hbar c}{16\pi\alpha_G^A} \hat{R}^{[\alpha\beta]}{}_{ab} \hat{R}_{[\alpha\beta]}{}^{ab}$$

$$- \frac{\hbar c}{16\pi\alpha_G^S} \hat{R}^{(\alpha\beta)}{}_{ab} \hat{R}_{(\alpha\beta)}{}^{ab} - \frac{\hbar c}{16\pi\alpha_G^T} \hat{R}^{\alpha}{}_{\alpha ab} \hat{R}^{\beta}{}_{\beta}{}^{ab} . \qquad (19)$$

These extra constants correspond to the arbitrariness in the choice of group metric.

I would regard a metric-connection theory with a gravitational Lagrangian of the form (19) as a Yang-Mills theory of gravity. I do not know whether such a theory even has a Newtonian limit. If it does, some combination of the coupling constants, L_1, L_1' and L_2, should be related to the Planck length, $L = (G\hbar/c^3)^{\frac{1}{2}}$. I interpret the coupling constants, α_G^A, α_G^S and α_G^T as gravitational fine structure constants by analogy with the electromagnetic fine structure constant in the Maxwell Lagrangian (II. 1.32).

A weaker assumption than (19) is that the gravitational Lagrangian is minimally constructed from ∇g, Q and $\hat{R}$. This means that the gravitational Lagrangian only depends on $\Gamma^{\alpha}{}_{\beta a}$ and the derivatives of $g_{\alpha\beta}$ and $\theta^{\alpha}{}_{a}$ through $\nabla_a g_{\alpha\beta}$, $Q^{\alpha}{}_{ab}$ and $\hat{R}^{\alpha}{}_{\beta ab}$ and a finite number of their derivatives:

$$L_G(\, g,\ldots,\partial^{(i)}g,\ \theta,\ldots,\partial^{(j)}\theta,\ \Gamma,\ldots,\partial^{(k)}\Gamma\,)$$

$$= L_G(\, g,\ \nabla g,\ldots,\nabla^{(p)}\nabla g,\ \theta,\ Q,\ldots,\nabla^{(q)}Q,\ \hat{R},\ldots,\nabla^{(r)}\hat{R}\,). \qquad (20)$$

I do not assume that the gravitational Lagrangian has the form (19) nor even the form (20).

I finally discuss the Yang-Mills analogy from the alternate viewpoint that the "curvatures" corresponding to $g_{\alpha\beta}$ and $\theta^{\alpha}{}_{a}$ are the non-covariant quantities, $\partial_a g_{\alpha\beta}$ and $c^{\alpha}{}_{ab}$ (rather than the tensors $\nabla_a g_{\alpha\beta}$ and $Q^{\alpha}{}_{ab}$). From this point of view, one is tempted to conclude that the gravitational Lagrangian most analogous to the Yang-Mills Lagrangian is the same as (19) except that the first three terms are replaced by

$$- s\,\frac{\hbar c}{16\pi L_1^{\,2}}\, g^{ab}\, g^{\alpha\gamma}\, g^{\beta\delta}(\partial_a g_{\alpha\beta})(\partial_b g_{\gamma\delta})$$

$$- s\,\frac{\hbar c}{16\pi L_1'^{\,2}}\, g^{ab}\, g^{\alpha\beta}\, g^{\gamma\delta}(\partial_a g_{\alpha\beta})(\partial_b g_{\gamma\delta})$$

$$- s\,\frac{\hbar c}{16\pi L_2^{\,2}}\, g_{\alpha\beta}\, g^{ac}\, g^{bd}\, c^{\alpha}{}_{ab}\, c^{\beta}{}_{cd}\,, \qquad (21)$$

However, the quantitiy (21) is not a scalar, and, used as a Lagrangian, it does not lead to tensorial field equations. The scalar closest to (21) is the scalar curvature of the Christoffel connection, $\tilde{R}$, which differs by the addition of a divergence from a quantity which is quadratic in $\partial_a g_{\alpha\beta}$ and $c^{\alpha}{}_{ab}$. Thus one is led to take the gravitational Lagrangian as

$$L_G = -s\frac{\hbar c}{16\pi L^2}\tilde{R} - \frac{\hbar c}{16\pi\alpha_G^A}\hat{R}^{[\alpha\beta]}{}_{ab}\,\hat{R}_{[\alpha\beta]}{}^{ab}$$

$$-\frac{\hbar c}{16\pi\alpha_G^S}\hat{R}^{(\alpha\beta)}{}_{ab}\,\hat{R}_{(\alpha\beta)}{}^{ab} - \frac{\hbar c}{16\pi\alpha_G^T}\hat{R}^{\alpha}{}_{\alpha ab}\,\hat{R}^{\beta\,ab}{}_{\beta}\ . \qquad (22)$$

However, at least for the case when the full connection is metric compatible, (so that $\hat{R}^{(\alpha\beta)}{}_{ab} = 0$,) this Lagrangian leads to separate conservation of spin and orbital angular momentum (as shown in Section V.3c), which is undesirable.

The next closest gravitational Lagrangian is obtained by replacing $\tilde{R}$ by $\hat{R}$:

$$L_G = -s\frac{\hbar c}{16\pi L^2}\hat{R} - \frac{\hbar c}{16\pi\alpha_G^A}\hat{R}^{[\alpha\beta]}{}_{ab}\,\hat{R}_{[\alpha\beta]}{}^{ab}$$

$$-\frac{\hbar c}{16\pi\alpha_G^S}\hat{R}^{(\alpha\beta)}{}_{ab}\,\hat{R}_{(\alpha\beta)}{}^{ab} - \frac{\hbar c}{16\pi\alpha_G^T}\hat{R}^{\alpha}{}_{\alpha ab}\,\hat{R}^{\beta\,ab}{}_{\beta}\ . \qquad (23)$$

In the metric-compatible case, this is the Lagrangian which I study in Chapter VI, and which Fairchild [1977] regards as the Lagrangian for a "Yang-Mills theory of gravity." Comparing Lagrangians (19) and (23), I choose to regard (19) as the better Yang-Mills Lagrangian for gravity.